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The evolution of finite-amplitude wavetrains in plane channel flow

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Contents

We consider a viscous incompressible fluid flow driven between two parallel plates by a constant pressure gradient. The flow is at a finite Reynolds number, with an $O(1)$ disturbance in the form of a travelling wave. A phase equation approach is used to discuss the evolution of slowly varying fully nonlinear two-dimensional wavetrains. We consider uniform wavetrains in detail, showing that the development of a wavenumber perturbation is governed by the Burgers equation in most cases. The wavenumber perturbation theory, constructed by using the phase equation approach for a uniform wavetrain, is shown to be distinct from an amplitude perturbation expansion about the periodic flow. In fact, we show that the amplitude equation contains only linear terms and is simply the heat equation. We review, briefly, the well-known dynamics of the Burgers equation, which imply that both shock structures and finite-time singularities of the wavenumber perturbation can occur with respect to the slow scales. Numerical computations have been performed to identify areas of the {wavenumber, Reynolds number, energy} neutral surface for which each of these possibilities can occur. We note that the evolution equations will break down under certain circumstances, in particular for a weakly nonlinear secondary flow. Finally, we extend the theory to three dimensions and discuss the limit of a weak spanwise dependence for uniform wavetrains, showing that two functions are required to describe the evolution. These unknowns are a phase and a pressure function which satisfy a pair of linearly coupled partial differential equations. The results obtained from applying the same analysis to the fully three-dimensional problem are included as an appendix.

Keywords: parallel shear flow; nonlinear stability; phase equation methods

1. Introduction

Fully nonlinear travelling wave solutions in plane Poiseuille flow (PPF) have been discussed by many authors in recent years (Zahn *et al*. 1974; Herbert 1976; Pugh & Saffman 1988), with emphasis placed upon the secondary instability problem. Instability of the finite-amplitude two-dimensional periodic flow to three-dimensional infinitesimal disturbances has been put forward traditionally as an explanation of transition. Orszag & Patera (1983) have also suggested that, even at Reynolds numbers below the subcritical minimum for nonlinear two-dimensional solutions, the time-scale for ultimate decay of the disturbance is sufficiently long for a small three-dimensional perturbation to grow strongly.

Subsequent work on superharmonic instability by Pugh & Saffman (1988) has shown that a more complex structure is to be expected than a simple stability transition at the Reynolds number limit point of the neutral surface. They show that parametrization of the problem is important and that bifurcations to quasi-periodic flows exist at points on the upper branch of the neutral surface, leaving open the possibility of such finite-amplitude solutions existing at a Reynolds number lower than the subcritical minimum for the periodic flow.

It should be noted that other approaches have been discussed that do not rely on the existence of finite-amplitude two-dimensional travelling waves. Although the possibility of resonant growth caused by linear mechanisms has been recognized for some time, it is only through recent work (Gustavsson 1981, 1991; Gustavsson & Hultgren 1980; Butler & Farrell 1992; Trefethen *et al*. 1992) concerning the initial value approach that the large amplification involved in three-dimensional problems has been revealed. The term 'bypass' has been attached to these methods since they do not follow the more traditional idea of transition arising from two-dimensional Tollmien–Schlichting waves with three-dimensional effects appearing only at a secondary instability stage.

The predictions of a linear stability analysis applied to plane Poiseuille flow have been confirmed experimentally by Nishioka *et al*. (1975). The experiments of Nishioka *et al*. followed similar discussions of flows in rectangular channels (see, for example, Kao & Park 1970; Davies & White 1928), however, as shown by Davis & White, the aspect ratio can have a significant effect on the critical Reynolds numbers obtained. By using a larger width-to-depth ratio (27.4 was used, whereas Kao & Park used a ratio of 8), together with reducing background noise to very low levels, Nishioka *et al*. managed to obtain good agreement between experimental data and the theoretically predicted critical Reynolds number of approximately 6000. In fact, Nishioka *et al*. note that it was possible to maintain a laminar flow in the test section for Reynolds numbers of up to 8000; however, this must obviously be interpreted in terms of the length of the channel. If a larger channel were used in the experiments, then these growing modes would eventually attain an amplitude above the threshold required to trigger transition.

All the experiments concerning plane Poiseuille flow confirm its subcritical nature, with transition from a laminar flow, in a sufficiently 'noisy' environment, being found at much lower Reynolds numbers than are predicted by a linear stability analysis (Alavyoon *et al*. (1986) give a value of 1100, based on centreline velocity and half-channel width). As noted above, the subcritical transition from a laminar state has been discussed theoretically both in terms of the secondary instability problem for nonlinear two-dimensional waves and for bypass mechanisms, which avoid the primary instability.

Returning to the experiments of Nishioka *et al.*, which introduced an artificial disturbance via a vibrating ribbon technique into an otherwise relatively noise-free flow, the growth of a two-dimensional instability wave to a threshold amplitude was noted before transition. In a later investigation, Kozlov & Ramazanov (1984) used a similar experimental set-up to investigate the effects of three-dimensional disturbances on the two-dimensional flow. They observed that when the two-dimensional wave had attained an amplitude above some threshold value it went through a breakdown process that involved three-dimensional structures referred to as Λ-vortices. Earlier, Klebanoff *et al*. (1962) also discussed this process in the context of three-dimensional disturbances to a Tollmien–Schlichting wave field in a flat-plate boundary layer.

In this discussion we return to the finite-amplitude two-dimensional travelling wave solutions described at the beginning of this section, and subsequently develop an evolution equation for a phase instability of the $O(1)$ flow. Since these nonlinear solutions are used frequently in some areas of both theory and computation, it seems sensible to try and discover something of their stability and evolution. We show, in fact, that uniform wavetrains will not be observed under certain classes of initial condition as they are susceptible to slow scale effects, with the wavenumber developing both singularities and shock structures after a finite time in the slow scale.

The method we use to determine the evolution equation is based upon the phase equation technique applied by numerous authors to Bénard convection problems with $O(1)$ amplitudes (for example, Newell *et al*. 1993). These same methods have been applied to wave problems by Howard & Kopell (1977), Whitham (1974), subsequently applied explicitly to the Ginzburg–Landau equation by Bernoff (1988) and used in an investigation of boundary layer instability theory by Hall (1995).

A detailed discussion of the phase equation method is given by Hall (1995) and Bernoff (1988), and is not repeated here. The essential idea is that we have a finite-amplitude wavetrain solution, which is locally periodic in space and time, allowing wavenumber and frequency to be functions varying on appropriate slow scales. The resulting equations of motion can then be rewritten in terms of the new scales and a phase function that is related to the wavenumber and frequency, through which it also satisfies a conservation equation. An expansion in terms of the slow-scale parameter will now yield a leading-order system that is a nonlinear eigenvalue problem. This relationship determines the local frequency of the wavetrain as a function of local wavenumber, Reynolds number (and indirectly amplitude), yielding both supercritical and subcritical equilibria for PPF. The next order problem will then provide a linear inhomogeneous system that determines the frequency correction term through a solvability condition; this technique also allows for continuation to higher orders. Now, since the wavetrain evolves according to the phase-conservation equation, we can, by expanding appropriately, give a slow-scale asymptotic approximation to the evolution equation.

In § 2 of this discussion, the above technique is applied explicitly to the finite Reynolds number *two*-dimensional PPF problem. In § 2 *a* we discuss the implications of the phase equation theory when applied to the stability of a *uniform* two-dimensional wavetrain. We show that the stability of a small wavenumber perturbation is governed by the Burgers equation,

$$\Delta_\tau + \Delta\Delta_\zeta = \pm\Delta_{\zeta\zeta}, \tag{1.1}$$

for $O(1)$ problems that correspond to distinct points on either the upper or lower branch of the neutral surface and away from the linear neutral curve. In § 3 we discuss the stability of the $O(1)$ flow to an amplitude perturbation, showing that, for these length-scales, nonlinear terms are not introduced and that the amplitude equation is simply the heat transfer equation. Section 4 provides a short description of the numerical methods involved in the stability calculations, presenting results that are consistent with those of other authors (Herbert 1976; Pugh & Saffman 1988). Computational results, which show the behaviour of the viscous diffusion term from the Burgers equation, are presented for differing leading-order problems. Section 5 returns to the analysis of a uniform wavetrain and briefly discusses how the theory breaks down for weakly nonlinear secondary flow solutions. In § 6 we redevelop the phase equation theory for a three-dimensional problem and consider the stability of wavenumber perturbations in the limit of a weak spanwise dependence. Finally, in § 7 we discuss the implications and future extensions of the work.

2. Formulation of the phase equation approach

We wish to consider a finite-amplitude solution to the plane Poiseuille flow (PPF) problem, in the form of a travelling wave, then allow for a slow modulation on the new scales,

$$X = \delta x \quad \text{and} \quad T = \delta t. \tag{2.1}$$

We now use the methods presented in Howard & Kopell (1977), Bernoff (1988), and applied to asymptotic suction boundary layer flow by Hall (1995). This analysis follows closely the finite Reynolds number case of Hall (1995), except for a few differences associated with a pressure eigenfunction term (discussed later). We first introduce a phase function, $\Theta(X,T) = \delta\theta(x,t)$, which allows a definition of the local frequency and wavenumber as

$$\alpha = \frac{\partial\Theta}{\partial X}, \quad \Omega = -\frac{\partial\Theta}{\partial T}, \tag{2.2}$$

where $\alpha = \alpha(X,T)$ and $\Omega = \Omega(X,T)$ are allowed to vary on the slow scales. Thus the partial derivatives transform as

$$\frac{\partial}{\partial x} \to \alpha\frac{\partial}{\partial\theta} + \delta\frac{\partial}{\partial X}, \quad \frac{\partial}{\partial t} \to -\Omega\frac{\partial}{\partial\theta} + \delta\frac{\partial}{\partial T}, \tag{2.3}$$

and a conservation of phase condition must be satisfied,

$$\frac{\partial\alpha}{\partial T} + \frac{\partial\Omega}{\partial X} = 0. \tag{2.4}$$

We can now develop a perturbation scheme about the fully nonlinear leading-order solution by introducing a slow scale expansion of the stream function,

$$\psi = \hat{\psi}_0 + \delta\psi_1 + \cdots, \tag{2.5}$$

which forces a similar expansion for the frequency,

$$\Omega = \Omega_0 + \delta\Omega_1 + \cdots . \tag{2.6}$$

Here $\hat{\psi}_0 = \bar{\Psi} + \psi_0$, with $\bar{\Psi} = y - \frac{1}{3}y^3$ the basic flow potential, plus a leading-order spatially periodic flow. We shall use the vorticity equation formulation to describe the flow,

$$\frac{\partial(\nabla^2\psi)}{\partial t} + \frac{\partial(\nabla^2\psi, \psi)}{\partial(x,y)} - \frac{1}{Re}\nabla^4\psi = 0, \tag{2.7}$$

with a Reynolds number, Re, defined as $\bar{U}^* h/\nu$, where $\bar{U}^*$ is the centerline velocity, $2h$ the plate separation and ν the kinematic viscosity. Thus at leading order, using the above expansions, we obtain

$$-\Omega_0\hat{\nabla}^2\hat{\psi}_{0\theta} + \alpha\hat{\psi}_{0y}\hat{\nabla}^2\hat{\psi}_{0\theta} - \alpha\hat{\psi}_{0\theta}\hat{\nabla}^2\hat{\psi}_{0y} - (1/Re)\hat{\nabla}^4\hat{\psi}_0 = 0, \tag{2.8}$$

with subscripts $\{\theta, y, \dots\}$ denoting the respective derivatives, where it is unambiguous to do so, and

$$\hat{\nabla}^2 \equiv \alpha^2\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial y^2}. \tag{2.9}$$

The boundary conditions are simply impermeability and no-slip at the parallel boundaries. In terms of the stream function (at this order) this gives

$$\hat{\psi}_{0\theta} = \hat{\psi}_{0y} = 0 \quad \text{at } y = \pm 1. \tag{2.10}$$

Similar situations have been discussed previously by Herbert (1976) and Pugh & Saffman (1988), and this nonlinear eigenvalue problem can be solved by using the same numerical techniques. Here we have an eigenrelation

$$\Omega_0 = \Omega_0(\alpha, Re), \tag{2.11}$$

which defines a 'neutral surface' in $\{\alpha, Re, \text{amplitude}\}$ parameter space. A schematic representation of the neutral surface is shown in figure 1, in which E denotes some energy measure of the two-dimensional wave. We shall return to the problem of solving (2.8) numerically in § 4 *a*, where we shall present some cross-sections of the form illustrated in figure 1.

Now at next order, after some rearrangement, we obtain

$$L\{\psi_1\} = \left[\Omega_0(2\alpha\hat{\psi}_{0\theta\theta\alpha} + \hat{\psi}_{0\theta\theta}) + \frac{\partial\Omega_0}{\partial\alpha}\frac{\partial\hat{\nabla}^2\hat{\psi}_0}{\partial\alpha} + \alpha\frac{\partial(\hat{\psi}_0, 2\alpha\hat{\psi}_{0\theta\alpha} + \hat{\psi}_{0\theta})}{\partial(\theta, y)} + \frac{\partial(\hat{\psi}_0, \hat{\nabla}^2\hat{\psi}_0)}{\partial(\alpha, y)} \right.$$
$$\left. + \frac{1}{Re}(4\alpha\hat{\nabla}^2\hat{\psi}_{0\theta\alpha} + 4\alpha^2\hat{\psi}_{0\theta\theta\theta} + 2\hat{\nabla}^2\hat{\psi}_{0\theta})\right]\frac{\partial\alpha}{\partial X} + \Omega_1\hat{\nabla}^2\hat{\psi}_{0\theta}, \tag{2.12}$$

with boundary conditions $\psi_{1y} = \psi_{1\theta} = 0$ at $y = \pm 1$. To obtain the above form, (2.12), we have introduced the operator L defined as

$$L \equiv -\Omega_0\hat{\nabla}^2\frac{\partial}{\partial\theta} + \alpha\hat{\psi}_{0y}\hat{\nabla}^2\frac{\partial}{\partial\theta} + \alpha\hat{\nabla}^2\hat{\psi}_{0\theta}\frac{\partial}{\partial y} - \alpha\hat{\psi}_{0\theta}\hat{\nabla}^2\frac{\partial}{\partial y} - \alpha\hat{\nabla}^2\hat{\psi}_{0y}\frac{\partial}{\partial\theta} - \frac{1}{Re}\hat{\nabla}^4, \tag{2.13}$$

and the $\partial/\partial T$ term has been replaced by using the conservation of phase as

$$\frac{\partial}{\partial T} \to -\frac{\partial}{\partial\alpha}(\Omega_0 + \delta\Omega_1 + \cdots)\frac{\partial\alpha}{\partial X}\frac{\partial}{\partial\alpha}. \tag{2.14}$$

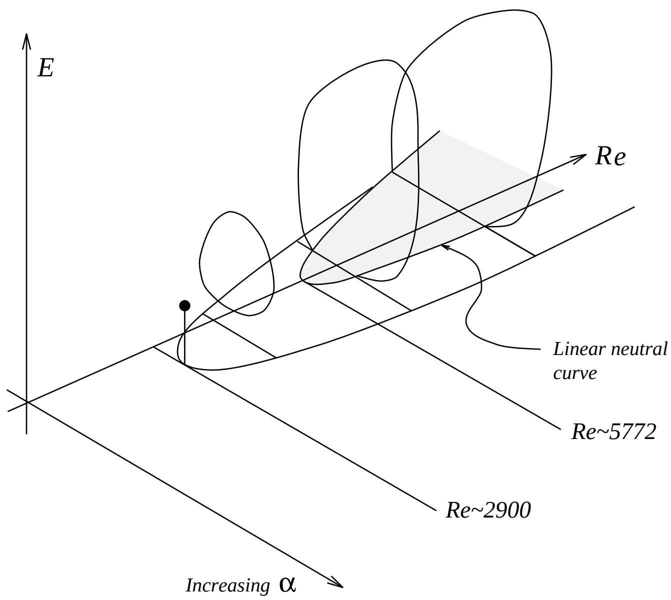

Figure 1. A schematic representation of the neutral surface.

We now consider the mean flow, which from the Navier–Stokes equation is governed by

$$\frac{1}{Re}\frac{\partial^2 u_{00}}{\partial y^2} = \sum_{n=1}^{\infty}\left(v_{0n}\frac{\partial u_{0n}^{(c)}}{\partial y} + v_{0n}^{(c)}\frac{\partial u_{0n}}{\partial y}\right) + \frac{\partial q_{-1}}{\partial X}. \tag{2.15}$$

Here we have expanded the velocity field and pressure as

$$\underline{\boldsymbol{u}} = \hat{\underline{\boldsymbol{u}}}_0(X,T,\theta,y) + \delta\underline{\boldsymbol{u}}_1(X,T,\theta,y) + \cdots, \tag{2.16}$$

$$p = [GX + q_{-1}(X,T)]\delta^{-1} + [p_0(X,T,\theta,y) + q_0(X,T)] + \cdots, \tag{2.17}$$

with

$$\hat{\underline{\boldsymbol{u}}}_0 = (\bar{U},0)^{\mathrm{T}} + \underline{\boldsymbol{u}}_0(X,T,\theta,y), \quad \underline{\boldsymbol{u}}_0 = (u_{00},v_{00})^{\mathrm{T}}, \tag{2.18}$$

$v_{00} \equiv 0$ from continuity, and

$$\underline{\boldsymbol{u}}_0 = \underline{\boldsymbol{u}}_{00} + \sum_{n=1}^{\infty}\{\underline{\boldsymbol{u}}_{0n}\mathrm{e}^{\mathrm{i}n\theta} + \underline{\boldsymbol{u}}_{0n}^{(c)}\mathrm{e}^{-\mathrm{i}n\theta}\}, \quad \underline{\boldsymbol{u}}_{0n} = (u_{0n},v_{0n})^{\mathrm{T}}. \tag{2.19}$$

In the above expansions, we have used $\bar{U} = 1 - y^2$ to denote the non-dimensionalized basic flow, $G = -2/Re$ is the basic driving pressure gradient and the superscript '(c)' denotes a complex conjugation. At next order we obtain

$$\frac{1}{Re}\frac{\partial^2 u_{10}}{\partial y^2} = \frac{\partial p_{00}}{\partial X} - \frac{\partial \Omega_0}{\partial \alpha}\frac{\partial u_{00}}{\partial \alpha}\alpha_X + \bar{U}\frac{\partial u_{00}}{\partial X} + \frac{\partial \bar{U}}{\partial y}v_{10} + \sum_{n=-\infty}^{\infty}\left(u_{0n}\frac{\partial u_{0n}^{(c)}}{\partial X} + v_{0n}\frac{\partial u_{1n}^{(c)}}{\partial y} + v_{1n}\frac{\partial u_{0n}^{(c)}}{\partial y}\right) + \frac{\partial q_0}{\partial X}, \tag{2.20}$$

where p_{00} is introduced from the expansion

$$p_0 = p_{00} + \sum_{n=1}^{\infty} \{p_{0n} \mathrm{e}^{\mathrm{i}n\theta} + p_{0n}^{(\mathrm{c})} \mathrm{e}^{-\mathrm{i}n\theta}\}, \tag{2.21}$$

and, from the $O(\delta^0)$ y momentum equation, satisfies

$$\frac{\partial p_{00}}{\partial y} = -\sum_{n=-\infty}^{\infty} \frac{\partial v_{0n} v_{0n}^{(\mathrm{c})}}{\partial y}. \tag{2.22}$$

In (2.20) we have also introduced

$$\underline{\boldsymbol{u}}_1 = \underline{\boldsymbol{u}}_{10} + \sum_{n=1}^{\infty} \{\underline{\boldsymbol{u}}_{1n} \mathrm{e}^{\mathrm{i}n\theta} + \underline{\boldsymbol{u}}_{1n}^{(\mathrm{c})} \mathrm{e}^{-\mathrm{i}n\theta}\}, \quad \underline{\boldsymbol{u}}_{1n} = (u_{1n}, v_{1n})^{\mathrm{T}}, \tag{2.23}$$

and the summation terms in (2.20) and (2.22) are performed with negative subscripts denoting a complex conjugation.

The equation of continuity at this order, $O(\delta)$, is

$$\alpha \frac{\partial u_1}{\partial \theta} + \frac{\partial u_0}{\partial X} + \frac{\partial v_1}{\partial y} = 0, \tag{2.24}$$

which, when considering the mean-flow terms only, reduces to

$$\frac{\partial v_{10}}{\partial y} = -\frac{\partial u_{00}}{\partial X}, \tag{2.25}$$

a first-order equation for the mean-flow correction to v, required to satisfy the impermeability conditions at *both* walls. This difficulty was anticipated earlier and is resolved by the introduction of a further, slow scale-dependent, pressure expansion

$$q_{-1}(X,T)\delta^{-1} + q_0(X,T) + \cdots, \tag{2.26}$$

producing the extra term $\partial q_{-1}/\partial X$ in (2.15) which is chosen to satisfy (2.25). Thus, for impermeability at both boundaries, we must satisfy

$$\int_{-1}^{+1} \frac{\partial u_{00}}{\partial X} \, \mathrm{d}y = 0, \tag{2.27}$$

which fixes the streamwise flux through the channel and hence determines q_{-1} as a function of X at given T. This is equivalent to solving the vorticity equation for the mean-flow correction with the same boundary conditions $\psi_{00} = \partial\psi_{00}/\partial y = 0$ at $y = \pm 1$, where ψ_{00} is the part of the stream function having zero mean with respect to the phase, θ.

Now we have an $O(\delta^0)$ problem that can be solved numerically to give $\Omega_0(\alpha, Re)$ and an expansion in terms of δ giving a further system (2.12), which is used to compute Ω_1 at a given neutral surface point. The homogeneous form of (2.12) is solved by $\psi_{0\theta}$ (corresponding to the existence of a translationally invariant solution, since any arbitrary constant may be added to the phase) and so Ω_1 is determined by a solvability condition at $O(\delta)$. The form of (2.12) is

$$\alpha \frac{\partial}{\partial \theta}(A\underline{\boldsymbol{q}}) + \frac{\partial}{\partial y}(B\underline{\boldsymbol{q}}) + C\underline{\boldsymbol{q}} = \underline{\boldsymbol{H}}, \tag{2.28}$$

so premultiplying by the adjoint vector, $\underline{\boldsymbol{r}}^{\mathrm{T}} = (r_1, \ldots, r_6)$, and integrating by parts gives

$$-\alpha \frac{\partial}{\partial \theta}(A^{\mathrm{T}}\underline{\boldsymbol{r}}) - \frac{\partial}{\partial y}(B^{\mathrm{T}}\underline{\boldsymbol{r}}) + C^{\mathrm{T}}\underline{\boldsymbol{r}} = \underline{\boldsymbol{0}} \tag{2.29}$$

as the adjoint equation for the homogeneous form of (2.28), with

$$\underline{\boldsymbol{q}} = (\psi_1, \alpha\psi_{1\theta}, \psi_{1y}, \hat{\nabla}^2\psi_1, \alpha\hat{\nabla}^2\psi_{1\theta}, \hat{\nabla}^2\psi_{1y})^{\mathrm{T}}, \tag{2.30}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(1/Re) & 0 \end{bmatrix}, \tag{2.31}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(1/Re) \end{bmatrix} \tag{2.32}$$

and

$$C = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -\hat{\nabla}^2\hat{\psi}_{0y} & \alpha\hat{\nabla}^2\hat{\psi}_{0\theta} & 0 & \hat{\psi}_{0y} - (\Omega_0/\alpha) & -\alpha\hat{\psi}_{0\theta} \end{bmatrix}. \tag{2.33}$$

The boundary conditions are also determined by the above process and are easily shown to be

$$r_5 = r_6 = 0 \quad \text{at } y = \pm 1, \tag{2.34}$$

plus a periodicity condition. Thus the solvability condition is

$$\int_{\theta=0}^{2\pi} \int_{y=-1}^{+1} r_6 H_6 \,\mathrm{d}y\,\mathrm{d}\theta = 0, \tag{2.35}$$

giving

$$\Omega_1 = \frac{\partial \alpha}{\partial X}\Phi(\alpha), \tag{2.36}$$

where

$$\Phi(\alpha) = \left\{ \int_{\theta=0}^{2\pi} \int_{y=-1}^{+1} M(\theta, y) r_6 \,\mathrm{d}y\,\mathrm{d}\theta \right\} \left\{ \int_{\theta=0}^{2\pi} \int_{y=-1}^{+1} -r_6 \hat{\nabla}^2\psi_{0\theta} \,\mathrm{d}y\,\mathrm{d}\theta \right\}^{-1}, \tag{2.37}$$

with M used to denote the term in the square brackets in equation (2.12) and $\underline{\boldsymbol{H}} = (0, \ldots, 0, H_6)^{\mathrm{T}}$.

Given Ω_1 we can now write down the phase conservation condition, up to $O(\delta)$, in the form

$$\frac{\partial\alpha}{\partial T}+\frac{\partial\Omega_0}{\partial\alpha}\frac{\partial\alpha}{\partial X}=\delta\frac{\partial}{\partial X}(-\Omega_1)+\cdots. \tag{2.38}$$

This is an evolution equation for $\alpha=\alpha(X,T)$, which we could *in principle* continue to any order through this expansion scheme; rather than solving this directly, we shall restrict our attention to a somewhat simpler problem.

(*a*) *Stability of uniform wavetrains*

Here we wish to discuss the stability of a fully nonlinear uniform wavetrain to a small slow-scale wavenumber perturbation. Hence, following the above expansion scheme, we obtain an $O(\delta^0)$ problem with a phase function of leading-order form,

$$\theta_0=\alpha_0 x-\Omega_0(\alpha_0)t, \tag{2.39}$$

where $\{\alpha_0, Re\}$ defines a point upon the neutral surface at $O(1)$ disturbance energy, with associated frequency Ω_0. If we now perturb the wavenumber of this uniform solution with a slowly varying function Δ,

$$\alpha=\alpha_0+\Delta(X,T), \tag{2.40}$$

and perform the above analysis, the evolution equation reduces to

$$\frac{\partial}{\partial T}\Delta(X,T)+\frac{\partial\Omega_0}{\partial\alpha}\frac{\partial}{\partial X}\Delta(X,T)+\delta\frac{\partial\Omega_1}{\partial X}+\delta^2\frac{\partial\Omega_2}{\partial X}+\cdots=0. \tag{2.41}$$

Now we introduce a Taylor series expansion for Ω_i about the uniform solution, and a transformation of the streamwise coordinate to a new frame of reference, moving with speed $\Omega_0'(\alpha_0)$, to obtain

$$\eta\frac{\partial\Delta}{\partial\tau}+\frac{\partial^2\Omega_0(\alpha_0)}{\partial\alpha^2}\Delta\frac{\partial\Delta}{\partial\xi}+\delta\Phi(\alpha_0)\frac{\partial^2\Delta}{\partial\xi^2}=O\left(\delta\Delta^2\frac{\partial\Delta}{\partial\xi},\delta\Delta\frac{\partial^2\Delta}{\partial\xi^2},\delta\frac{\partial\Delta^2}{\partial\xi}\right). \tag{2.42}$$

Here we have defined τ, ξ and $\Phi(\alpha)$ by

$$\tau=\eta T, \tag{2.43}$$

$$\xi=X-\Omega_0'(\alpha_0)T, \tag{2.44}$$

so that

$$\frac{\partial}{\partial T}\to\eta\frac{\partial}{\partial\tau}-\Omega_0'(\alpha_0)\frac{\partial}{\partial\xi}, \tag{2.45}$$

$$\frac{\partial}{\partial X}\to\frac{\partial}{\partial\xi}. \tag{2.46}$$

We therefore obtain a leading-order balance for $\eta=\delta$ and $\Delta\sim O(\delta)$.

A rescaling of $\bar{\Delta}=\Omega_0''(\alpha_0)\Delta$ will reduce this evolution equation to the standard form for the Burgers equation, namely (after dropping the bar notation)

$$\Delta_\tau+\Delta\Delta_\xi=-\Phi(\alpha_0)\Delta_{\xi\xi}, \tag{2.47}$$

which has well-known properties. This approach again follows that given by both Hall (1995) and Bernoff (1988), where the Burgers equation has been derived by applying this technique to both the asymptotic suction boundary layer and the Ginzburg–Landau equation. Whitham (1974) has discussed the dynamics of this equation in detail; it has an exact solution via the Cole–Hopf transformation,

$$\Delta = 2\Phi(\alpha_0)\frac{\partial}{\partial\xi}\log(\Lambda), \tag{2.48}$$

which removes the nonlinear term, reducing the evolution equation to the form of the heat diffusion equation,

$$\Lambda_\tau = -\Phi(\alpha_0)\Lambda_{\xi\xi} + C(\tau)\Lambda, \tag{2.49}$$

where $C(\tau)$ is set to be identically zero since this merely corresponds to a scaling of the dependent variable. Now for the initial value problem, with known $\Delta(\xi,0) = F(\xi)$, it is possible to obtain the analytical solution

$$\Lambda(\xi,\tau) = \frac{1}{\sqrt{-4\pi\Phi(\alpha_0)\tau}}\int_{-\infty}^{+\infty}\exp\left\{\frac{(\xi-\eta)^2}{4\Phi(\alpha_0)\tau} + \frac{1}{2\Phi(\alpha_0)}\int_0^\eta F(\bar\eta)\,\mathrm{d}\bar\eta\right\}\mathrm{d}\eta, \tag{2.50}$$

through an application of Laplace transform methods. Therefore the solution is

$$\Delta(\xi,\tau) = \left\{\int_{-\infty}^{+\infty}\frac{\xi-\eta}{\tau}\mathrm{e}^{D/[2\Phi(\alpha_0)]}\,\mathrm{d}\eta\right\}\left\{\int_{-\infty}^{+\infty}\mathrm{e}^{D/[2\Phi(\alpha_0)]}\,\mathrm{d}\eta\right\}^{-1}, \tag{2.51}$$

where

$$D = \int_0^\eta F(\bar\eta)\,\mathrm{d}\bar\eta + \frac{(\xi-\eta)^2}{2t}. \tag{2.52}$$

This can support both shock structures and singularities at finite times. For a positive diffusive term on the right of (2.47), $\Phi(\alpha_0) < 0$, we have a bounded solution that will decay for a localized/periodic disturbance. For a negative right-hand side, $\Phi(\alpha_0) > 0$, the solution is diffusively unstable and will become singular at finite time, indicating that the slow variation assumption is no longer appropriate and a return to the full equations of motion is required. Weak shock structures are discussed by Bernoff (1988) who notes that for a small monotonic wavenumber variation such that

$$\lim_{x\to-\infty}\alpha = \alpha_- \tag{2.53}$$

and

$$\lim_{x\to+\infty}\alpha = \alpha_+, \tag{2.54}$$

the Burgers equation applies as a leading-order form for the evolution equation if

$$\Delta\alpha = \alpha_+ - \alpha_- \ll 1. \tag{2.55}$$

Here we also require that the unmodulated wavetrain corresponds to a distinct point on either the upper or lower branch of the neutral surface at $O(1)$ amplitude. Now this wavenumber variation will eventually become concentrated into a weak shock structure of width $O(1/\Delta\alpha)$ in the unscaled streamwise coordinate if

$$\Omega_0'(\alpha_0)(\alpha_+ - \alpha_-) < 0, \tag{2.56}$$

moving with speed

$$c = (\Omega(\alpha_+) - \Omega(\alpha_-))/\Delta\alpha \tag{2.57}$$

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(a discretized form of the group velocity). If this variation in wavenumber ($\Delta\alpha$) increases, then the slow-scale assumptions are eventually lost (as with the finite-time singularity case) and the evolution of the wave system is governed by the full equations, namely Navier–Stokes.

In this case (stability of uniform wavetrains) we should also note that the leading-order problem is simplified since there is no slow-scale dependency for the wavenumber. Thus the effect of the extra pressure term $\partial q_{-1}/\partial X$ (now a constant) is to induce an additional parabolic velocity profile into the mean-flow correction, and this corresponds to a scaling of the Reynolds number at fixed amplitude/wavenumber. So the condition (2.27), which fixes the flux through the channel, effectively determines a unique parametrization of the problem (as discussed by Pugh & Saffman (1988)) in a self-consistent manner.

3. An amplitude perturbation approach

We now show how the phase equation method described previously is distinct from a more typical amplitude perturbation approach. In this method we solve the same leading-order problem for the uniform wavetrain, but it is now perturbed by an eigenfunction with a slowly varying amplitude $B = B(X, T)$. Again we introduce the slow scales,

$$X = \delta x, \tag{3.1}$$

and

$$T = \delta t, \tag{3.2}$$

together with

$$\hat{X} = (X - c_{\mathrm{g}} T), \tag{3.3}$$

a new moving coordinate system, and a further time-scale,

$$\hat{T} = \delta T. \tag{3.4}$$

This slower time-scale is known from the previous section but could otherwise be determined from the final solvability condition. Here c_{g} is a group velocity and, given values for α and Ω, we can expand in terms of δ and a phase variable $\theta = \alpha x - \Omega t$. Now, seeking a solution analogous to that in Hall (1995), we expand the stream function as

$$\psi = \hat{\psi}_0 + \delta\psi_1 + \cdots, \tag{3.5}$$

where

$$\hat{\psi}_0 = \bar{\Psi} + \psi_0, \tag{3.6}$$

$$\psi_1 = B(\hat{X}, \hat{T})\alpha \frac{\partial \psi_0}{\partial \theta}, \tag{3.7}$$

with $\bar{\Psi}$ corresponding to the basic parallel flow, and

$$\psi_n = \frac{[B(\hat{X}, \hat{T})\alpha]^n}{n!} \frac{\partial^n \psi_0}{\partial \theta^n} + \tilde{\psi}_n, \quad n \geqslant 2. \tag{3.8}$$

We now return to the vorticity equation (2.7) and substitute the above expansions along with

$$\frac{\partial}{\partial x} \to \alpha \frac{\partial}{\partial \theta} + \delta \frac{\partial}{\partial \hat{X}}, \tag{3.9}$$

$$\frac{\partial}{\partial t} \to -\Omega \frac{\partial}{\partial \theta} - \delta c_{\mathrm{g}} \frac{\partial}{\partial \hat{X}} + \delta^2 \frac{\partial}{\partial \hat{T}}, \tag{3.10}$$

$$\hat{\nabla}^2 \equiv \frac{\partial^2}{\partial y^2} + \alpha^2 \frac{\partial^2}{\partial \theta^2}, \tag{3.11}$$

to give the following leading order the form of the vorticity equation:

$$O(\delta^0) : -\Omega \hat{\nabla}^2 \hat{\psi}_{0\theta} + \alpha \hat{\psi}_{0y} \hat{\nabla}^2 \hat{\psi}_{0\theta} - \alpha \hat{\psi}_{0\theta} \hat{\nabla}^2 \hat{\psi}_{0y} - (1/Re) \hat{\nabla}^4 \hat{\psi}_0 = 0. \tag{3.12}$$

At next order we obtain

$$O(\delta) : -\Omega \hat{\nabla}^2 \psi_{1\theta} + \alpha \hat{\psi}_{0y} \hat{\nabla}^2 \psi_{1\theta} + \alpha \psi_{1y} \hat{\nabla}^2 \hat{\psi}_{0\theta} - \alpha \hat{\psi}_{0\theta} \hat{\nabla}^2 \psi_{1y} - \alpha \psi_{1\theta} \hat{\nabla}^2 \hat{\psi}_{0y} - (1/Re) \hat{\nabla}^4 \psi_1 = 0, \tag{3.13}$$

with solution (3.7), which is an amplitude perturbation of the underlying periodic flow; in this formulation ψ_0 is *independent* of the slow scales $\hat{X}, \hat{T}$. The required group velocity is now determined from the next-order system, which can be rearranged more clearly as

$$L_{O(\delta)}\{\tilde{\psi}_2\} = \alpha[2\alpha\Omega \hat{\psi}_{0\theta\theta\theta} + c_{\mathrm{g}} \hat{\nabla}^2 \hat{\psi}_{0\theta} - \hat{\psi}_{0y}(2\alpha^2 \hat{\psi}_{0\theta\theta\theta} + \hat{\nabla}^2 \hat{\psi}_{0\theta}) + \hat{\psi}_{0\theta}(2\alpha^2 \hat{\psi}_{0\theta\theta y} + \hat{\nabla}^2 \hat{\psi}_{0y}) + (4\alpha/Re) \hat{\nabla}^2 \hat{\psi}_{0\theta\theta}] B_{\hat{X}}, \tag{3.14}$$

with the $L_{O(\delta)}$ operator defined by the $O(\delta)$ equation, (3.13), and once terms proportional to B^2 have been eliminated by taking $\partial/\partial\theta$ of (3.12). We observe that $\tilde{\psi}_2 = \alpha B_{\hat{X}} \psi_{0\alpha}$ and, as expected, c_{g} corresponds to $\partial\Omega_0(\alpha)/\partial\alpha$ in the phase equation approach; this follows by taking $\partial/\partial\alpha$ of (3.12). Note that an additional multiple of the homogeneous solution to $\tilde{\psi}_2$ will not alter the solvability condition at next order, but will contribute to an amplitude equation at higher order.

We also must remember that the additional pressure term (q_{-1}, discussed in § 2) is still required in the leading-order mean-flow problem. Obviously, a similar pressure term, $\partial q_{-1}/\partial\alpha$, is necessary at $O(\delta^2)$, but it is not until $O(\delta^3)$ that the condition determining q_{-1} is obtained. Now continuity of mass, at $O(\delta^3)$, requires the same constant mass-flow condition to be satisfied for impermeability of the boundaries. Thus we determine q_{-1}, appearing in the leading-order problem, in the same manner as discussed in the phase equation approach.

The same process can be repeated for the problem at next order, which becomes

$$L_{O(\delta)}\{\tilde{\psi}_3\} = BB_{\hat{X}}\{K(\theta, y)\} + B_{\hat{T}}\{-\alpha \hat{\nabla}^2 \psi_{0\theta}\} + B_{\hat{X}\hat{X}}\{M(\theta, y)\alpha\}, \tag{3.15}$$

after again eliminating the B^3 terms by taking $\partial^2/\partial\theta^2$ of (3.12). The M expression, in (3.15), is as defined previously in the phase equation analysis with $\{\Omega_0, \partial_\alpha \Omega_0\}$

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replaced by $\{\Omega, c_{\mathrm{g}}\}$, and K is given by

$$K = 2\alpha^3 \Omega \psi_{0\theta\theta\theta\theta} + c_{\mathrm{g}} \alpha^2 \hat{\nabla}^2 \psi_{0\theta\theta} + 2\alpha^4 \frac{\partial(\psi_0, \psi_{0\theta\theta\theta})}{\partial(\theta, y)} + \alpha^2 \frac{\partial(\psi_0, \hat{\nabla}^2 \psi_{0\theta})}{\partial(\theta, y)} + \alpha^3 \frac{\partial(\psi_{0\theta}, \hat{\nabla}^2 \psi_{0\alpha})}{\partial(\theta, y)} + 2\alpha^4 \frac{\partial(\psi_{0\theta}, \psi_{0\theta\theta})}{\partial(\theta, y)} + \alpha^2 \frac{\partial(\psi_{0\theta}, \hat{\nabla}^2 \psi_0)}{\partial(\theta, y)} + \alpha^3 \frac{\partial(\psi_{0\alpha}, \hat{\nabla}^2 \psi_{0\theta})}{\partial(\theta, y)} - \frac{1}{Re} \hat{\nabla}^2 \psi_{0\theta\theta\theta}. \tag{3.16}$$

Although at first sight (3.15) appears to be the Burgers equation, reproduced through an amplitude perturbation approach, it does in fact reduce to a simpler form once we have observed that a particular solution is available to remove the nonlinear term from the solvability condition,

$$L_{O(\delta)} \left\{ \alpha^2 \frac{\partial \psi_{0\alpha}}{\partial \theta} \right\} = K(\theta, y). \tag{3.17}$$

Similarly by an inductive method we can show that a rescaling does not reintroduce nonlinear terms (which will be of the form $B^n \partial B / \partial \hat{X}$) into the amplitude equation, since we can develop a general particular solution,

$$\frac{\alpha^{n+1}}{n!} \frac{\partial^n \psi_{0\alpha}}{\partial \theta^n}. \tag{3.18}$$

If we now follow the same method outlined in § 2, we can obtain a solvability condition at this order, namely

$$B_{\hat{T}} \int_y \int_\theta -r_6 \hat{\nabla}^2 \psi_{0\theta} \, \mathrm{d}\theta \, \mathrm{d}y + B_{\hat{X}\hat{X}} \int_y \int_\theta M(\theta, y) r_6 \, \mathrm{d}\theta \, \mathrm{d}y = 0. \tag{3.19}$$

This is the heat transfer equation; the solution is characterized by the sign of the diffusive term, given by

$$\left\{ \int_{y=-1}^{+1} \int_{\theta=0}^{2\pi} M(\theta, y) r_6 \, \mathrm{d}\theta \, \mathrm{d}y \right\} \left\{ \int_{y=-1}^{+1} \int_{\theta=0}^{2\pi} -r_6 \hat{\nabla}^2 \hat{\psi}_0 \theta \, \mathrm{d}\theta \mathrm{d}y \right\}^{-1} \equiv \Phi(\alpha) \tag{3.20}$$

in the previous notation. So we note that these amplitude perturbations are of less interest than the corresponding phase instabilities; since they are governed by the heat equation, the solutions will simply decay exponentially to zero or grow becoming singular in a finite time. Thus we have stable decaying solutions for $\Phi(\alpha) < 0$ but when $\Phi(\alpha) > 0$ we must at some stage return to the full evolution equations to determine the development as higher-order spatial and temporal derivatives are reintroduced.

4. Numerical methods

(*a*) *The leading-order problem*

In solving the leading-order form of the vorticity equation we look for solutions that expand as

$$\hat{\psi}_0 = \bar{\Psi} + \sum_{n=-\infty}^{\infty} \psi_{0n} \mathrm{e}^{\mathrm{i}n\theta}, \tag{4.1}$$

where $\hat{\psi}_0$ is the leading-order term in the expansion of the stream function ψ, (2.5), and $\bar{U} \equiv D\bar{\Psi}$. The vorticity equation (2.7) reduces, after substitution of the above form, to

$$\frac{1}{Re}(D^2 - n^2\alpha^2)^2\psi_{0n} - \mathrm{i}\alpha n\left\{\left[\bar{U} - \frac{\Omega_0}{\alpha}\right](D^2 - n^2\alpha^2)\psi_{0n} - \psi_{0n}D^2\bar{U}\right\}$$
$$+\mathrm{i}\alpha\sum_{m=-\infty}^{\infty}\left\{(n-m)\psi_{0n-m}(D^3 - m^2\alpha^2 D)\psi_{0m} - mD\psi_{0n-m}(D^2 - m^2\alpha^2)\psi_{0m}\right\} = 0, \tag{4.2}$$

for $n = 0, \pm 1, \pm 2, \ldots$ with boundary conditions $\psi_{0n} = D\psi_{0n} = 0$, where $D \equiv \mathrm{d}/\mathrm{d}y$. Now we solve this with a truncation of the Fourier modes and a Chebyshev expansion in the y direction for each harmonic,

$$\psi_{0n}(y) = \tfrac{1}{2}a_{n0} + \sum_{r=1}^{\infty} a_{nr}T_r(y), \tag{4.3}$$

along with $\bar{U} = \frac{1}{2}(1 - T_2)$. This formulation has been applied previously by Herbert (1976) to plane Poiseuille flow with a constant pressure gradient rather than the constant mass-flux condition we apply in this case.

If the harmonics are truncated at N_{h}, and the Chebyshev series at N_{c}, this yields a numerical problem that can be solved by using Lanczo's τ method with $N_{\mathrm{h}}(N_{\mathrm{c}} + 7)$ nonlinear equations plus a coupled mean-flow problem. We also note that the computational task can be simplified by assuming the symmetry condition,

$$\psi_n(y) = (-1)^{n+1}\psi_n(-y), \tag{4.4}$$

together with requiring that the solution to (4.2) is real, so that

$$\psi_{-n} = \psi_{+n}^{(\mathrm{c})}. \tag{4.5}$$

The τ method is essentially equivalent to determining the higher Chebyshev coefficients through the boundary conditions; however, we can only replace two of the dynamical equations with boundary conditions, and therefore have to retain at least one τ element. Various methods were investigated for the solution of these simultaneous nonlinear equations, and finally a Newton iteration technique was found to provide the best convergence over large amplitude ranges. We write the nonlinear system of equations as

$$\underline{\boldsymbol{f}}(\underline{\boldsymbol{x}}) = \underline{\boldsymbol{0}}, \tag{4.6}$$

where

$$\underline{\boldsymbol{x}} = (\Omega_0, a_{11}, a_{12}, \ldots, a_{1N_{\mathrm{c}}}, a_{20}, a_{21}, \ldots, a_{2N_{\mathrm{c}}}, a_{N_{\mathrm{h}}0}, \ldots, a_{N_{\mathrm{h}}N_{\mathrm{c}}})^{\mathrm{T}}, \tag{4.7}$$

and at each iteration level solve

$$J\underline{\boldsymbol{z}} = -\underline{\boldsymbol{f}}(\underline{\boldsymbol{x}}_k), \tag{4.8}$$

to give

$$\underline{\boldsymbol{x}}_{k+1} = \underline{\boldsymbol{x}}_k + \underline{\boldsymbol{z}}, \tag{4.9}$$

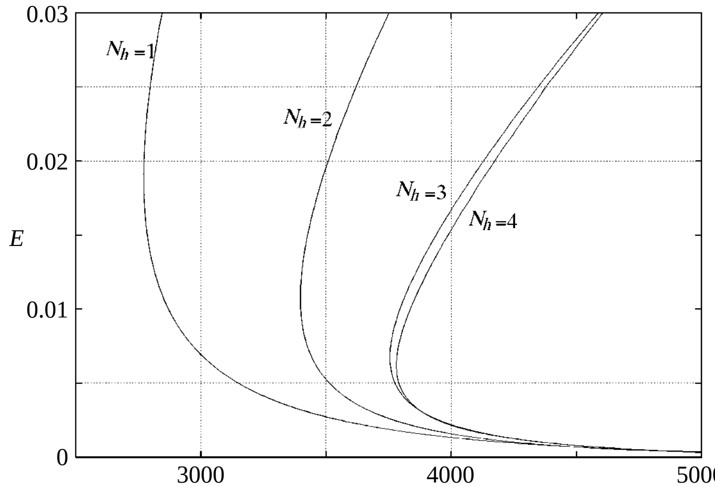

Figure 2. A cross-section of the neutral surface at $\alpha = 1.08$, with $N_{\rm h} = 1, \ldots, 4$.

where J is the Jacobian

$$J \equiv \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_m} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \dfrac{\partial f_m}{\partial x_2} & \cdots & \dfrac{\partial f_m}{\partial x_m} \end{bmatrix}. \tag{4.10}$$

The number of unknowns in this solution procedure can be effectively halved by applying the symmetry assumptions discussed above. From this nonlinear eigenvalue problem we determine the relationship between Ω_0 and $\{\alpha, Re\}$. To do this we specify an amplitude of disturbance by choosing a value for the first Chebyshev coefficient, a_{10}, in the expansion of the fundamental mode. Without loss of generality, we can assume that this amplitude measure is real since this corresponds to a unique determination of the phase. Here the initial guess for a solution of the system, at a general neutral surface point, is derived from an interpolation of previous 'nearby' solutions. Since a value for one of the Chebyshev coefficients is specified we must replace it by some other unknown in the Newton iteration scheme, typically the frequency.

As discussed in § 2, we must also iterate upon the additional pressure constant, $\partial q_{-1}/\partial_X$ in the mean-flow equation (2.15), to satisfy the constant mass-flux condition. At each level of the iteration scheme described above we compute $\underline{\boldsymbol{x}}_k$, then the corresponding mean-flow problem is solved directly, including the constant $\partial q_{-1}/\partial_X$, from (2.15). We continue the iteration, as described above, until some measure of convergence is satisfied, which in this case we choose to be that the frequency correction and largest change in Chebyshev coefficients are less than preset tolerances.

In figures 2, 3 and 4 we show typical cross-sections of the neutral surface at fixed wavenumbers of 1.08, 0.95 and $\{1.1, 0.9707\}$, respectively. The results of figure 4 (for $\alpha = 1.1$, $N_{\rm h} = 1$) are consistent with those obtained by Pugh & Saffman (1988). The

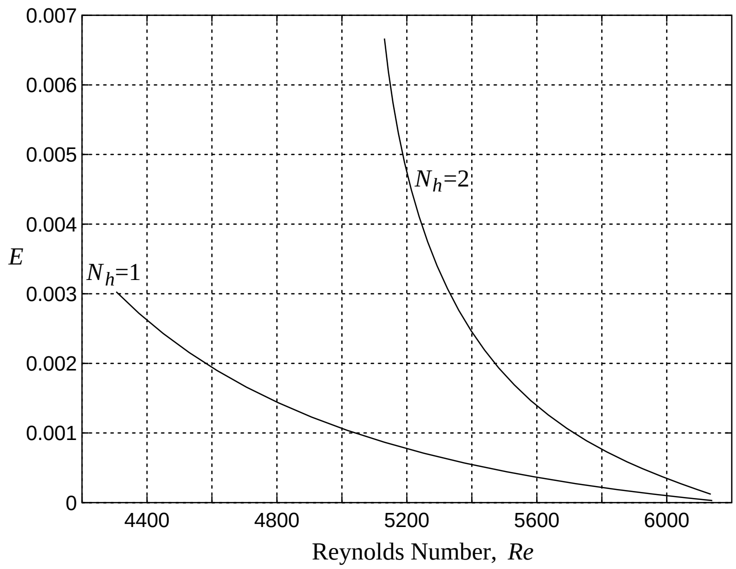

Figure 3. A cross-section of the neutral surface at $\alpha = 0.95$, with $N_{\mathrm{h}} = 1, 2$.

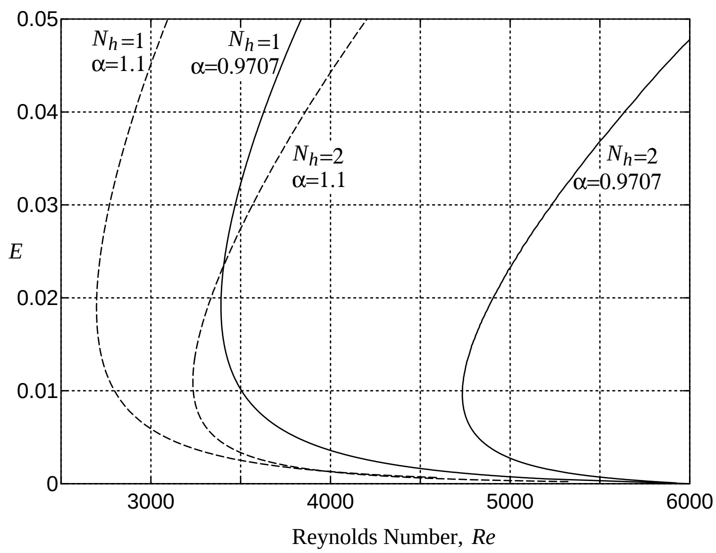

Figure 4. Cross-sections of the neutral surface at $\alpha = 0.9707, 1.1$, with $N_{\mathrm{h}} = 1, 2$.

energy of the two-dimensional wave is denoted by E in the figures, and defined by

$$E = \sum_{n=1}^{N_{\mathrm{h}}} E_n, \tag{4.11}$$

where

$$E_n = \frac{15}{8} \int_{-1}^{+1} (|D\psi_{0n}|^2 + |n\alpha\psi_{0n}|^2)\, \mathrm{d}y, \tag{4.12}$$

a normalized energy measure for each harmonic.

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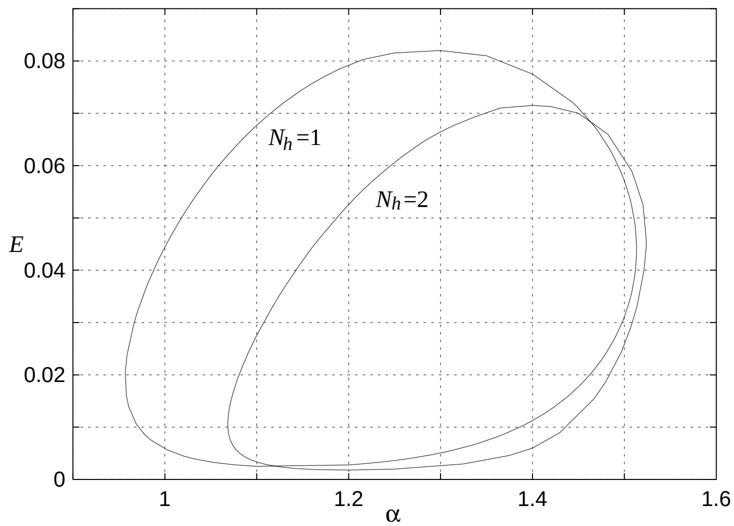

Figure 5. A cross-section of neutral surface at $Re = 3500$, with $N_{\rm h} = 1, 2$.

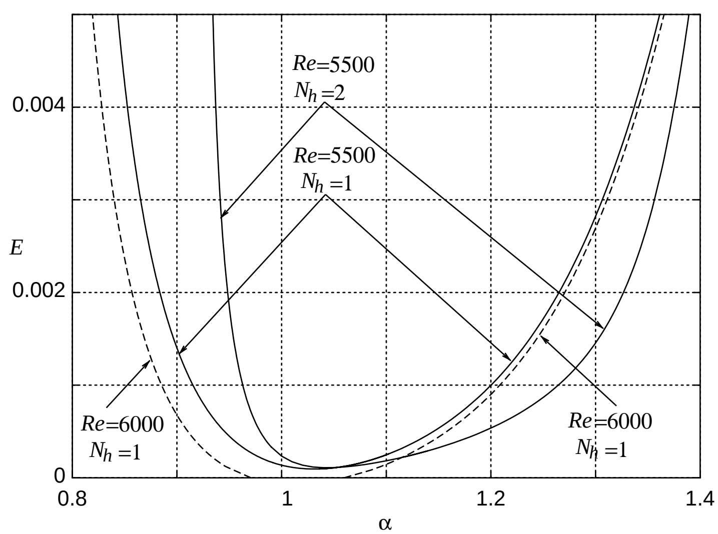

Figure 6. Cross-sections of the neutral surface at subcritical ($Re = 5500$) and supercritical ($Re = 6000$) Reynolds numbers.

Figure 5 displays the form of the cross-section for a fixed Reynolds number of $Re = 3500$. Since this Reynolds number is subcritical the neutral surface of figure 5 does not intersect the $E = 0$ plane, but defines a threshold energy above which equilibrium solutions can be located. In figure 6 we show the lower branch of the neutral surface at fixed Reynolds numbers of 5500 (subcritical) and 6000 (supercritical), these parameter values (together with those chosen for figures 2–4) will be used in the following computations of the coefficient Φ.

We found that $N_{\rm c} = 40$ was generally sufficient to give converged results for any chosen number of harmonics in this Reynolds number range. We note that, as shown in figure 5 (also figure 2), the overall effect of including higher harmonics is to shift the neutral surface to higher streamwise wavenumbers and to a slightly lower energy range. Although the inclusion of higher harmonics (i.e. $N_{\mathrm{h}} \geqslant 4$) still altered the mid-range amplitude results, the qualitative behaviour remained the same, as noted in Herbert's (1976) investigation of the slightly different problem of a constant pressure gradient flow.

(*b*) *The $O(\delta)$ problem*

To determine the sign of the diffusive coefficient on the right of the Burgers equation we need to solve the next order problem. In § 2 *a* the phase equation analysis of the uniform wavetrain problem showed that the frequency correction term was determined by the solvability condition

$$\begin{aligned} \Omega_1 = \frac{\partial \alpha}{\partial X}\left\{ \int_{\theta=0}^{2\pi}\int_{y=-1}^{+1} -\hat{\nabla}^2\hat{\psi}_{0\theta} r_6 \,\mathrm{d}y\,\mathrm{d}\theta \right\}^{-1} \\ \times \left\{ \int_{\theta=0}^{2\pi}\int_{y=-1}^{+1} \left[\Omega_0(2\alpha\hat{\psi}_{0\theta\theta\alpha} + \hat{\psi}_{0\theta\theta}) + \Omega_{0\alpha}\frac{\partial\hat{\nabla}^2\hat{\psi}_0}{\partial\alpha} + \alpha\frac{\partial(\hat{\psi}_0, 2\alpha\hat{\psi}_{0\theta\alpha} + \hat{\psi}_0\theta)}{\partial(\theta, y)} \right.\right. \\ \left.\left. + \frac{\partial(\hat{\psi}_0, \hat{\nabla}^2\hat{\psi}_0)}{\partial(\alpha, y)} + Re^{-1}\{4\alpha\hat{\nabla}^2\hat{\psi}_{0\theta\alpha} + 4\alpha^2\hat{\psi}_{0\theta\theta\theta} + 2\hat{\nabla}^2\hat{\psi}_{0\theta}\} \right] r_6 \,\mathrm{d}y\,\mathrm{d}\theta \right\}. \end{aligned} \tag{4.13}$$

Here $\hat{\psi}_0$ is the basic parallel flow plus two-dimensional periodic flow and r_6 solves the adjoint problem, which can be shown to be

$$\begin{aligned} -Re^{-1}\hat{\nabla}^4 r_6 - (\alpha\hat{\psi}_{0y} - \Omega_0)\hat{\nabla}^2 r_{6\theta} + \alpha\hat{\psi}_{0\theta}\hat{\nabla}^2 r_{6y} \\ - 2\alpha\hat{\psi}_{0y\theta}\left(\alpha^2\frac{\partial^2}{\partial\theta^2} - \frac{\partial^2}{\partial y^2}\right) r_6 + 2\alpha r_{6\theta y}\left(\alpha^2\frac{\partial^2}{\partial\theta^2} - \frac{\partial^2}{\partial y^2}\right)\hat{\psi}_0 = 0, \end{aligned} \tag{4.14}$$

with boundary conditions $r_6 = r_{6\theta} = 0$ at $y = \pm 1$. However, in our calculations we do not solve the linear homogeneous problem (4.14) but rather define a new composite nonlinear problem that is an inhomogeneous form of the order one system; this can be computed directly after the leading-order solution by using the same routine. We solve

$$N\{\tilde{\psi}_0; \tilde{\Omega}_0\} = \delta M(\theta, y) + O(\delta^2), \tag{4.15}$$

where the operator N and the new parameters are given by

$$N = L_{O(1)} + \delta L_{O(\delta)}, \tag{4.16}$$

$$\tilde{\Omega}_0 = \Omega_0 + \delta\Omega_1, \tag{4.17}$$

$$\tilde{\psi}_0 = \hat{\psi}_0 + \delta\psi_1, \tag{4.18}$$

with M and $L_{O(\cdot)}$ as defined previously. Hence, after solving the leading-order problem, we compute the inhomogeneous terms produced by M, then iterate on the amplitude of the new composite solution at fixed $\{\alpha, Re\}$. The frequency correction term is then determined by an application of the condition $\Omega_1 = \partial\tilde{\Omega}_0/\partial\delta$ at $\delta = 0$.

A point to note is that we divide the neutral surface into two sections, which are referred to as the upper and lower branches. We can see from (4.13) that we should expect a singular behaviour for the coefficient of the diffusive term as we approach

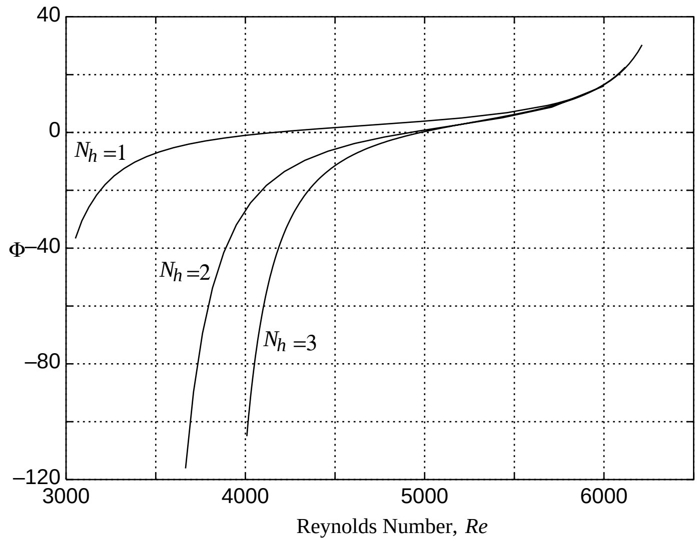

Figure 7. The variation of Φ at $\alpha = 1.08$ with $N_{\mathrm{h}} = 1, 2, 3$.

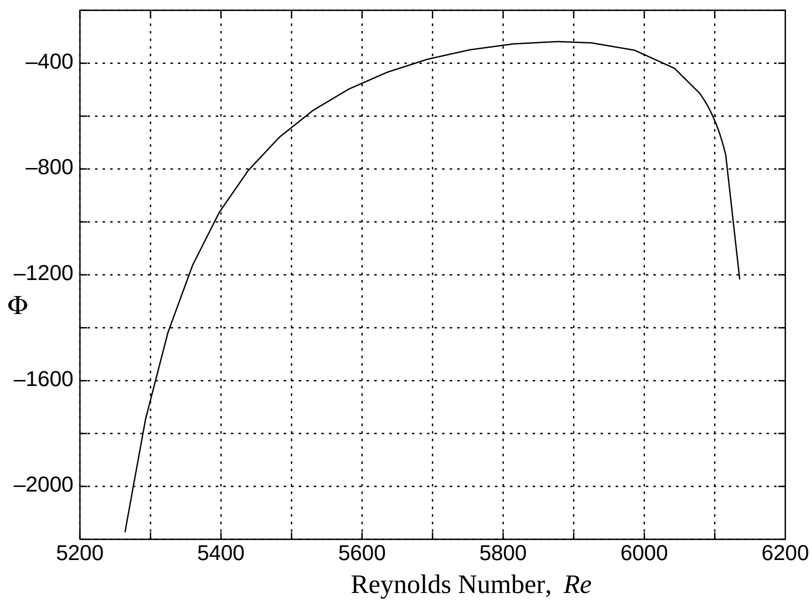

Figure 8. The variation of Φ at $\alpha = 0.95$ with $N_{\mathrm{h}} = 2$.

two other regions, namely where the lower branch connects with the $E = 0$ plane and where the upper and lower branches join at finite amplitude. The singularities are due to the way in which the amplitude and frequency vary with α at these points, and further work is required near these regions; we present some discussion of this limit in § 5.

We present, in figures 7 and 8, the behaviour of the diffusion coefficient in the evolution equation as the leading-order solution varies along the lower branches shown in figures 2 and 3. Since the evolution of a wavenumber perturbation to a uniform

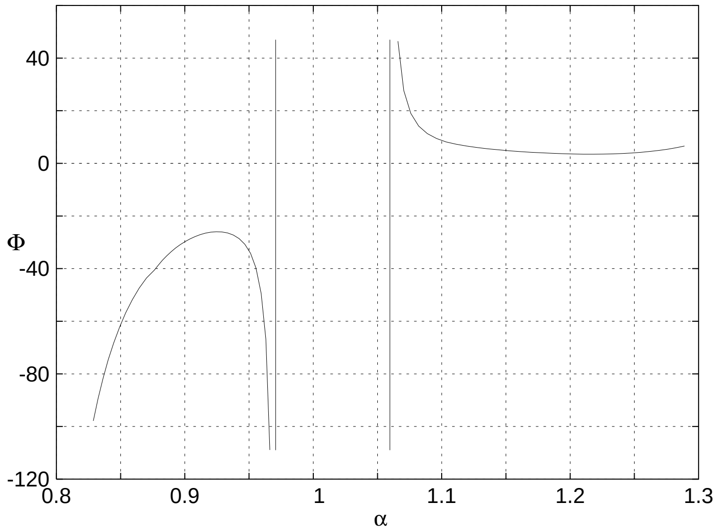

Figure 9. The variation of Φ at a supercritical Reynolds number ($Re = 6000$) with $N_{\rm h} = 1$.

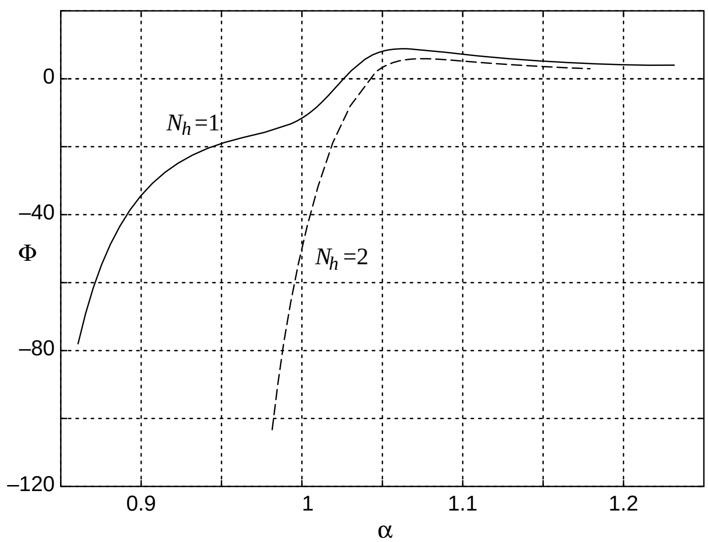

Figure 10. The variation of Φ at a subcritical Reynolds number ($Re = 5500$) with $N_{\rm h} = 1, 2$.

wavetrain (as discussed in § 2 *a*) is governed by the Burgers equation, (2.47), and the diffusive coefficient in (2.47) is $-\Phi$, we see that the solution is essentially characterized by the sign of Φ.

Since the linear neutral curve exists for each of the wavenumbers represented in figures 7 and 8 we find a singular behaviour of Φ at a Reynolds number for which the neutral surface intersects the $E = 0$ plane as discussed above.

We have carried out calculations (of Φ), with more harmonics, for a number of typical parameter values and found that such results have the same qualitative behaviour. In fact, the main effect of the higher harmonics is to alter the cross-sectional shape of the neutral surface as shown in figure 5, thus altering the position of the singularities

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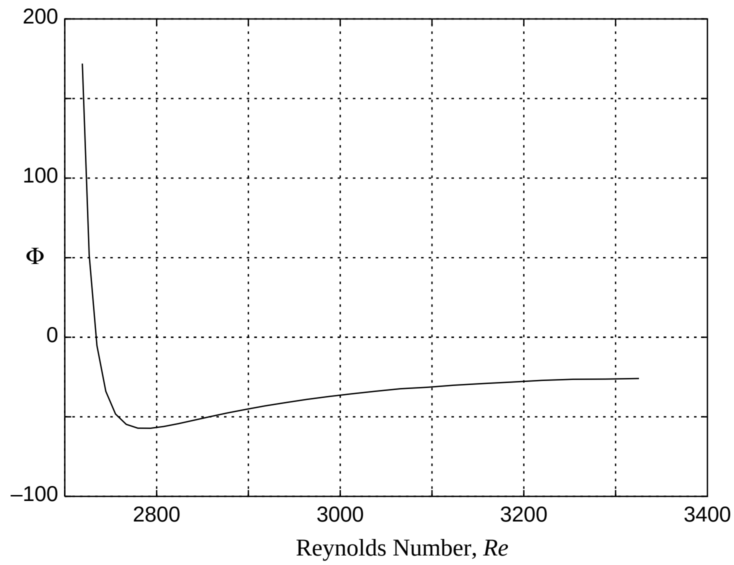

Figure 11. The variation of Φ for upper branch solutions at $\alpha = 1.1$ with $N_{\mathrm{h}} = 1$.

when evaluating Φ for varying Reynolds numbers, wavenumber or amplitude. This can result in a quantitative difference in the value of Φ computed for different $N_{\mathrm{h}} \leqslant 3$ (as seen in figure 7); however, since we are particularly interested only in the sign of Φ the inclusion of higher harmonics does not change the conclusion that both signs are possible in general. If we wished to compute the location of points on the neutral surface at which Φ changes sign, then obviously more harmonics would be required for accurate values.

In figures 9 and 10 we show the behaviour of Φ as a function of the wavenumber α for fixed Reynolds numbers; the vertical lines in figure 9 show the critical wavenumbers predicted by linear theory. These figures are computed for the sections of the neutral surface shown in figure 6, and thus the wavenumbers represented by the vertical lines in figure 9 are those at which the neutral surface for $Re = 6000$ (the dashed curve shown in figure 6) intersects the $E = 0$ plane (also see figure 1).

Finally, figures 11 and 12 show the behaviour of Φ for solutions that correspond to points upon the upper branch of the neutral surface at $\alpha = 1.1$, $N_{\mathrm{h}} = 1$ and $\alpha = 0.9707$, respectively.

5. The evolution equation as $E \to 0$

As noted in the previous section, the phase equation method is only valid for finite-amplitude periodic states; if we allow the leading-order solution to approach the linear limit then the coefficient Φ, in the Burgers equation, becomes singular. A simplistic approach would be to return to the uniform wavetrain problem and allow the $O(\delta^0)$ system to be a small-amplitude solution with a phase function of leading-order form

$$\theta_0 = \alpha_0 x - \Omega_0(\alpha_0)t. \tag{5.1}$$

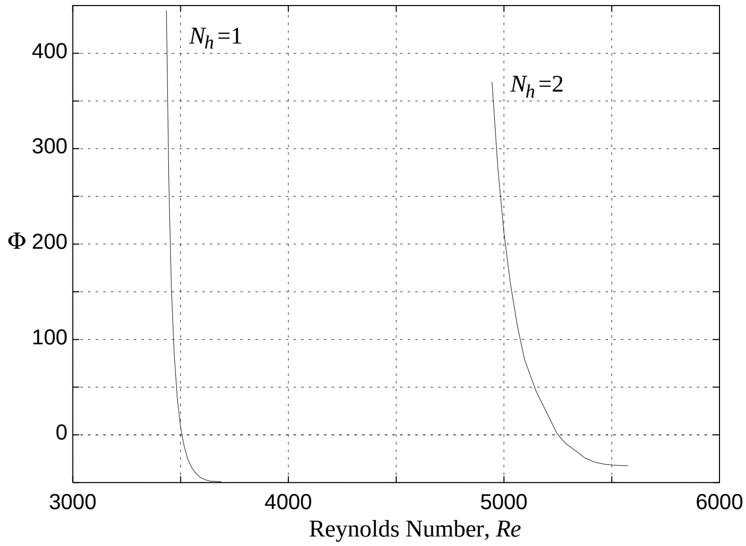

Figure 12. The variation of Φ for upper branch solutions at $\alpha = 0.9707$ with $N_\mathrm{h} = 1, 2$.

In this formulation, $\{\alpha_0, Re\}$ is a point on the lower branch of the neutral surface in the weakly nonlinear region with

$$\alpha_0 = \alpha_n + \tilde{\alpha}, \quad |\tilde{\alpha}| \ll 1, \tag{5.2}$$

where α_n is a wavenumber corresponding to a point on the linear neutral curve. If we perturb this solution with a slowly varying function Δ,

$$\alpha = \alpha_0 + \Delta(X, T), \tag{5.3}$$

then, as we approach the linear neutral curve, we know that the neutral surface has the scalings

$$A = O(|\alpha - \alpha_n|^{1/2}), \tag{5.4}$$

$$\Omega = \Omega_n + O(\alpha - \alpha_n), \tag{5.5}$$

for some amplitude measure A and where Ω_n is a frequency associated with the linear solution. The full form of the evolution equation is

$$\frac{\partial \Delta}{\partial T} + (\Omega_0(\alpha_0) + \Delta \Omega_0'(\alpha_0) + \cdots)\frac{\partial \Delta}{\partial X} = \delta \frac{\partial}{\partial X}\left\{\Phi(\alpha)\frac{\partial \Delta}{\partial X} + \delta \Omega_2(\alpha) + \cdots\right\}, \tag{5.6}$$

or, equivalently,

$$\eta \frac{\partial \Delta}{\partial \tau} + \Omega_0'(\alpha_0)\Delta \frac{\partial \Delta}{\partial \xi} + \left\{\Omega_0''(\alpha_0)\Delta^2 \frac{\partial \Delta}{\partial \xi} + \cdots\right\} = \delta \Phi(\alpha_0)\frac{\partial^2 \Delta}{\partial \xi^2}$$
$$+ \delta\left\{\Phi'(\alpha_0)\left(\frac{\partial \Delta}{\partial \xi}\right)^2 + \Phi'(\alpha_0)\Delta \frac{\partial^2 \Delta}{\partial \xi^2} + \cdots\right\} + \delta^2 \frac{\partial \Omega_2}{\partial \xi} + \cdots, \tag{5.7}$$

after applying the substitutions (2.43)–(2.44).

Therefore approaching the linear neutral curve may reintroduce other terms from the expression

$$\frac{\partial \Omega_1}{\partial \xi} = \frac{\partial \Phi}{\partial \alpha_0}\left(\frac{\partial \Delta}{\partial \xi}\right)^2 + \frac{\partial \Phi}{\partial \alpha_0}\Delta \frac{\partial^2 \Delta}{\partial \xi^2} + \cdots, \tag{5.8}$$

since applying the scalings (5.4)–(5.5) to equation (4.13) we see that

$$\Phi(\alpha_0) \sim 1/\tilde{\alpha}, \tag{5.9}$$

and the above terms can be of the same order of magnitude when

$$\eta \sim \Delta \sim \delta^{1/2} \sim \tilde{\alpha}. \tag{5.10}$$

This limit is obviously non-uniform; the phase equation theory, as presented in the previous sections, is aimed towards a treatment of fully nonlinear waves, but we see that the well-known theory concerning weakly nonlinear waves is not simply a small amplitude limiting case of the phase equation approach.

The reason for this difficulty in connecting the two approaches in the small amplitude limit is that, as we approach the linear neutral curve, the amplitude will no longer be determined explicitly from the leading-order eigenvalue problem (2.8). So rather than assuming the weakly nonlinear scalings (5.4)–(5.5) we need to introduce the amplitude of the two-dimensional wave into the analysis as an active parameter. We also note that the solvability requirements are complicated by taking the small amplitude limit (see Newell *et al*. 1993). It is the introduction of the amplitude into the theory that would allow us to satisfy these extra solvability conditions. This would result in two real equations that may then, presumably, be combined to recover a single complex evolution equation equivalent to that obtained by an application of a Stuart–Watson weakly nonlinear approach to the plane Poiseuille flow problem. Furthermore, in this limit, we no longer have a small parameter, δ, that is arbitrary; it will now be related to the magnitude of the wavenumber displacement from the linear neutral value, α_n.

Connection of the phase equation method to the appropriate weakly nonlinear theory proves difficult in general, indeed the analysis still remains incomplete for convection problems in which phase equation methods have been applied for some years. The recent works of Newell *et al*. (1993), Passot & Newell (1994) and Cross & Newell (1984) have discussed the points above and gone some way towards resolving the difficulties, although they restrict attention to much simplified model equations for convection (for example, the Swift–Hohenburg equation). In particular, Cross & Newell (1984) show how the evolution equation for such a model system can be matched with the Newell–Whitehead–Segel (NWS) equation, which was developed for convection problems with Rayleigh numbers approaching the linear critical value. In their analysis they demonstrate that the limiting forms of the phase evolution equation and the amplitude equation (required in this limit) form the imaginary and real parts of the NWS evolution equation respectively, hence they can be combined to match with previous results.

6. A three-dimensional phase equation theory

We now apply a generalized (following Howard & Kopell 1977) form of the two-dimensional phase equation method to a spanwise dependent problem. To achieve this we first introduce a further spanwise scale,

$$Z = \delta z, \tag{6.1}$$

and generalize the phase as a function of three variables, $\theta(x, z, t) = \Theta(X, Z, T)/\delta$, now defining a spanwise wavenumber, β, by

$$\beta = \Theta_Z. \tag{6.2}$$

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The consistency conditions in this approach are

$$\alpha_T + \Omega_X = 0, \tag{6.3}$$
$$\alpha_Z - \beta_X = 0, \tag{6.4}$$
$$\beta_T + \Omega_Z = 0, \tag{6.5}$$

where both wavenumbers and frequency are functions of the slow scales X, Z and T. We restrict attention to the stability of uniform wavetrains, considering a leading-order problem that is an oblique travelling wave solution at finite amplitude. We now perturb the wavenumbers as

$$\alpha = \alpha_0 + \delta \Delta_1(X, Z, T), \tag{6.6}$$
$$\beta = \beta_0 + \delta \Delta_2(X, Z, T), \tag{6.7}$$

and hence transform the partial derivatives in the following manner:

$$\frac{\partial}{\partial x} \to \alpha_0 \frac{\partial}{\partial \theta} + \delta\left(\Delta_1 \frac{\partial}{\partial \theta} + \frac{\partial}{\partial X}\right), \tag{6.8}$$
$$\frac{\partial}{\partial z} \to \beta_0 \frac{\partial}{\partial \theta} + \delta\left(\Delta_2 \frac{\partial}{\partial \theta} + \frac{\partial}{\partial Z}\right), \tag{6.9}$$
$$\frac{\partial}{\partial t} \to -\Omega_0 \frac{\partial}{\partial \theta} + \delta\left(-\Omega_1 \frac{\partial}{\partial \theta} + \frac{\partial}{\partial T}\right) + \cdots . \tag{6.10}$$

We also expand the velocity field and pressure function as

$$\underline{\boldsymbol{u}} = (1 - y^2, 0, 0)^{\mathrm{T}} + \underline{\boldsymbol{u}}_0(\theta, y) + \delta \underline{\boldsymbol{u}}_1(X, Z, T, \theta, y) + \cdots , \tag{6.11}$$
$$p = [-(2/Re)X + q_{-1}(X, Z)]\delta^{-1} + [p_0(\theta, y) + q_0(X, Z, T)] + \cdots , \tag{6.12}$$

where q_i are the additional pressure terms required to satisfy a constant mass flux through the channel.

Now as before we can expand the Navier–Stokes equations in terms of the slow-scale parameter, giving a leading-order system

$$-\Omega_0 \hat{u}_{0\theta} + \alpha_0 \hat{u}_0 \hat{u}_{0\theta} + v_0 \hat{u}_{0y} + \beta_0 w_0 \hat{u}_{0\theta} - \frac{1}{Re} \hat{\nabla}_3^2 \hat{u}_0 + \alpha_0 p_{0\theta} = -\frac{\partial q_{-1}}{\partial X}, \tag{6.13}$$
$$-\Omega_0 v_{0\theta} + \alpha_0 \hat{u}_0 v_{0\theta} + v_0 v_{0y} + \beta_0 w_0 v_{0\theta} - \frac{1}{Re} \hat{\nabla}_3^2 v_0 + p_{0y} = 0, \tag{6.14}$$
$$-\Omega_0 w_{0\theta} + \alpha_0 \hat{u}_0 w_{0\theta} + v_0 w_{0y} + \beta_0 w_0 w_{0\theta} - \frac{1}{Re} \hat{\nabla}_3^2 w_0 + \beta_0 p_{0\theta} = -\frac{\partial q_{-1}}{\partial Z}, \tag{6.15}$$
$$\alpha_0 \hat{u}_{0\theta} + v_{0y} + \beta_0 w_{0\theta} = 0, \tag{6.16}$$

$$\hat{\nabla}_3^2 \equiv (\alpha_0^2 + \beta_0^2) \frac{\partial^2}{\partial \theta^2} + \frac{\partial}{\partial y^2}, \tag{6.17}$$

where $\hat{u}_0 = \bar{U} + u_0$. At this order we let $q_{-1}(X, Z) = \kappa_1 X + \kappa_2 Z$ and choose κ_1, κ_2 to satisfy zero mass flux through the channel for the disturbance. Continuing further yields an $O(\delta)$ system of the form

$$L_u\{u_1,v_1,w_1,p_1\} = \Delta_1\left[-\hat{u}_0\hat{u}_{0\theta} - p_{0\theta} + \frac{2}{Re}\alpha_0\hat{u}_{0\theta\theta}\right] + \Delta_2\left[-w_0\hat{u}_{0\theta} + \frac{2}{Re}\beta_0\hat{u}_{0\theta\theta}\right] + \Omega_1\hat{u}_{0\theta} - \frac{\partial q_0}{\partial X}, \tag{6.18}$$

$$L_v\{u_1,v_1,w_1,p_1\} = \Delta_1\left[-\hat{u}_0 v_{0\theta} + \frac{2}{Re}\alpha_0 v_{0\theta\theta}\right] + \Delta_2\left[-w_0 v_{0\theta} + \frac{2}{Re}\beta_0 v_{0\theta\theta}\right] + \Omega_1 v_{0\theta}, \tag{6.19}$$

$$L_w\{u_1,v_1,w_1,p_1\} = \Delta_1\left[-\hat{u}_0 w_{0\theta} + \frac{2}{Re}\alpha_0 w_{0\theta\theta}\right] + \Delta_2\left[-w_0 w_{0\theta} - p_{0\theta} + \frac{2}{Re}\beta_0 w_{0\theta\theta}\right] + \Omega_1 w_{0\theta} - \frac{\partial q_0}{\partial Z}, \tag{6.20}$$

$$\alpha_0 u_{1\theta} + v_{1y} + \beta_0 w_{1\theta} = \Delta_1[-\hat{u}_{0\theta}] + \Delta_2[-w_{0\theta}], \tag{6.21}$$

where $L_u\{u_1,v_1,w_1,p_1\}$ is given by

$$L_u\{u_1,v_1,w_1,p_1\} \equiv -\Omega u_{1\theta} + \alpha_0\hat{u}_0 u_{1\theta} + \alpha_0 u_1\hat{u}_{0\theta} + v_0 u_{1y} + v_1\hat{u}_{0y} + \beta_0 w_0 u_{1\theta} + \beta_0 w_1\hat{u}_{0\theta} - (1/Re)\hat{\nabla}_3^2 u_1 + \alpha_0 p_{1\theta}, \tag{6.22}$$

with analogous forms for L_v and L_w.

After some work we can see that the solution is of the form

$$(u_1,v_1,w_1,p_1)^{\mathrm{T}} = \underline{\phi}_{11}\Delta_1(X,Z,T) + \underline{\phi}_{12}\Delta_2(X,Z,T) + \underline{\phi}_{13}\frac{\partial q_0}{\partial X}(X,Z,T) + \underline{\phi}_{14}\frac{\partial q_0}{\partial Z}(X,Z,T), \tag{6.23}$$

and the solvability condition determines Ω_1 as

$$\Omega_1 = \frac{\partial\Omega_0}{\partial\alpha_0}\Delta_1(X,Z,T) + \frac{\partial\Omega_0}{\partial\beta_0}\Delta_2(X,Z,T), \tag{6.24}$$

where we have used

$$\underline{\phi}_{11} = \frac{\partial\underline{\phi}_0}{\partial\alpha_0} - \frac{\partial\kappa_1}{\partial\alpha_0}\underline{\phi}_{13} - \frac{\partial\kappa_2}{\partial\alpha_0}\underline{\phi}_{14}, \tag{6.25}$$

$$\underline{\phi}_{12} = \frac{\partial\underline{\phi}_0}{\partial\beta_0} - \frac{\partial\kappa_1}{\partial\beta_0}\underline{\phi}_{13} - \frac{\partial\kappa_2}{\partial\beta_0}\underline{\phi}_{14}, \tag{6.26}$$

$$\underline{\phi}_{13} = \left.\frac{\partial\underline{\phi}_0}{\partial\kappa_1}\right|_{\{\alpha_0,\beta_0,\Omega_0\}\ \mathrm{fixed}}, \tag{6.27}$$

$$\underline{\phi}_{14} = \left.\frac{\partial\underline{\phi}_0}{\partial\kappa_2}\right|_{\{\alpha_0,\beta_0,\Omega_0\}\ \mathrm{fixed}}, \tag{6.28}$$

with $\underline{\phi}_0$ replaced by $\{u_0,v_0,w_0,p_0\}$.

In the usual two-dimensional iteration scheme (where Re is fixed), we specify $\{\alpha_0, \text{amplitude}\}$ and iterate on $\{\Omega_0,\kappa_1\}$. Now for given wavenumber and amplitude, we can think of Ω_0 as a function of κ_1 with the required frequency given by $\Omega_0(\kappa_1^*)$, where $\kappa_1 = \kappa_1^*$ is determined by the zero mass-flux condition for the disturbance. Obviously, we could equally fix $\{\alpha_0, \Omega_0\}$ and iterate on $\{\text{amplitude}, \kappa_1\}$ to define the amplitude as a function of the pressure constant. We can apply this same argument to the three-dimensional iteration scheme and so define the solutions $\{\underline{\phi}_{13}, \underline{\phi}_{14}\}$.

If we continue the expansion scheme to next order, we obtain a similar system of equations with inhomogeneous terms of the form $\{\Delta_1^2, \Delta_2^2, \Delta_{1X}, \Delta_{2X}, \Delta_1 q_{0X}, \dots\}$; we state the results for the fully three-dimensional problem in Appendix A, but to simplify this problem (and outline the basic method) we shall consider the weakly three-dimensional limit of $\beta_0 \to 0$. In this limit, the leading-order system will be the same two-dimensional problem computed in previous sections, but the solutions to the $O(\delta)$ equations are

$$(u_1, v_1, w_1, p_1)^{\mathrm{T}} = (u_{11}, v_{11}, 0, p_{11})^{\mathrm{T}} \Delta_1 + (0, 0, w_{12}, 0)^{\mathrm{T}} \Delta_2 + (u_{13}, v_{13}, 0, p_{13})^{\mathrm{T}} \frac{\partial q_0}{\partial X} + (0, 0, w_{14}, 0)^{\mathrm{T}} \frac{\partial q_0}{\partial Z}, \quad (6.29)$$

where

$$(u_{11}, v_{11}, p_{11})^{\mathrm{T}} = \frac{\partial}{\partial \alpha_0}(u_0, v_0, p_0)^{\mathrm{T}} - \frac{\partial \kappa_1}{\partial \alpha_0}(u_{13}, v_{13}, p_{13})^{\mathrm{T}}, \quad (6.30)$$

$$(u_{13}, v_{13}, p_{13})^{\mathrm{T}} = \frac{\partial}{\partial \kappa_1}(u_0, v_0, p_0)^{\mathrm{T}} \bigg|_{\{\alpha_0, \beta_0, \Omega_0\}\ \text{fixed}}, \quad (6.31)$$

$$w_{12} = \frac{\partial w_0}{\partial \beta_0} - \frac{\partial \kappa_2}{\partial \beta_0} w_{14}, \quad (6.32)$$

$$w_{14} = \frac{\partial w_0}{\partial \kappa_2}\bigg|_{\{\alpha_0, \beta_0, \Omega_0\}\ \text{fixed}}. \quad (6.33)$$

The solvability condition at this order reduces to

$$\Omega_1 = \frac{\partial \Omega_0}{\partial \alpha_0} \Delta_1(X, Z, T), \quad (6.34)$$

since for $\beta_0 \sim O(\gamma)$, $\gamma \ll 1$,

$$\left.\begin{aligned} \frac{\partial}{\partial \alpha_0}\{\hat{u}_0, v_0, p_0, \kappa_1\},\ \{p_{13}, u_{13}, v_{13}\} &\sim O(1), \\ \frac{\partial}{\partial \alpha_0}\{w_0, \kappa_2\},\ w_{13} &\sim O(\gamma), \\ \frac{\partial}{\partial \beta_0}\{\hat{u}_0, v_0, p_0, \kappa_1\},\ \{p_{14}, u_{14}, v_{14}\} &\sim O(\gamma), \\ \frac{\partial}{\partial \beta_0}\{w_0, \kappa_2\},\ w_{14} &\sim O(1). \end{aligned}\right\} \quad (6.35)$$

The group velocity obtained in this limit is equivalent to the two-dimensional case, as expected. Continuing the expansion scheme further, to $O(\delta^2)$, yields the following after some simplification:

$$\bar{L}_u\{u_2, v_2, p_2\} = \left[\frac{\partial \Omega_0}{\partial \alpha_0} u_{11\theta} - \hat{u}_0 u_{11\theta} - \alpha_0 u_{11} u_{11\theta} - u_{11}\hat{u}_{0\theta} - v_{11} u_{11y} - p_{11\theta} + \frac{1}{Re}(2\alpha_0 u_{11\theta\theta} + \hat{u}_{0\theta\theta})\right] \Delta_1^2 + \left[-w_{12}\hat{u}_{0\theta} + \frac{1}{Re}(\hat{u}_{0\theta\theta})\right]\Delta_2^2$$
$$+ [-\alpha_0 u_{13} u_{13\theta} - v_{13} u_{13y}]\left(\frac{\partial q_0}{\partial X}\right)^2 + [-w_{13}\hat{u}_{0\theta}]\Delta_2 \frac{\partial q_0}{\partial Z}$$
$$+ \left[\frac{\partial \Omega_0}{\partial \alpha_0} u_{13\theta} - \hat{u}_0 u_{13\theta} - \alpha_0 u_{11} u_{13\theta} - \alpha_0 u_{13} u_{11\theta} - u_{13}\hat{u}_{0\theta} - v_{11} u_{13y} - v_{13} u_{11y} - p_{13\theta} + \frac{1}{Re}(2\alpha_0 u_{13\theta\theta})\right]\Delta_1 \frac{\partial q_0}{\partial X}$$
$$+ \left[\frac{\partial \Omega_0}{\partial \alpha_0} u_{13} - \hat{u}_0 u_{13} - p_{13} + \frac{1}{Re}(2\alpha_0 u_{13\theta})\right]\frac{\partial^2 q_0}{\partial X^2}$$
$$+ \left[\frac{\partial \Omega_0}{\partial \alpha_0} u_{11} - \hat{u}_0 u_{11} - p_{11} + \frac{1}{Re}(2\alpha_0 u_{11\theta} + \hat{u}_{0\theta})\right]\frac{\partial \Delta_1}{\partial X}$$
$$+ \left[\frac{1}{Re}\hat{u}_{0\theta}\right]\frac{\partial \Delta_2}{\partial Z} + \Omega_2 \hat{u}_{0\theta} - \frac{\partial q_1}{\partial X}, \qquad (6.36)$$

with an analogous form for the v momentum equation and a decoupled w momentum problem. Continuity of mass at this order gives the further equation,

$$\alpha_0 \frac{\partial u_2}{\partial \theta} + \frac{\partial v_2}{\partial y} = -u_{11}\frac{\partial \Delta_1}{\partial X} - u_{13}\frac{\partial^2 q_0}{\partial X^2} - w_{12}\frac{\partial \Delta_2}{\partial Z} - w_{14}\frac{\partial^2 q_0}{\partial Z^2} - u_{11\theta}\Delta_1^2 - u_{13\theta}\Delta_1 \frac{\partial q_0}{\partial X} - w_{12\theta}\Delta_2^2 - w_{14\theta}\Delta_2 \frac{\partial q_0}{\partial Z}. \qquad (6.37)$$

We have used $\bar{L}_u$ (in (6.36)) to indicate the operator L_u, defined by (6.22), with $\beta_0 = 0$, and the boundary conditions for the systems discussed above are no-slip and impermeability at the fixed planes $y = \pm 1$. In this three-dimensional formulation, the additional pressure term, q_0, will contribute explicitly to the solvability condition for Ω_2; whereas in the previous theory this term was formally eliminated by using the stream function and vorticity equation approach. This is analogous to the work of Davey *et al*. (1974), concerning the weakly nonlinear evolution of three-dimensional disturbances in plane Poiseuille flow, where they found that a secular pressure term contributed to the final form of the evolution equation.

If we consider the continuity equation for those terms which have zero mean with respect to the phase variable θ, we obtain

$$\frac{\partial \bar{v}_2}{\partial y} = -\bar{u}_{11}\frac{\partial \Delta_1}{\partial X} - \bar{u}_{13}\frac{\partial^2 q_0}{\partial X^2} - \bar{w}_{12}\frac{\partial \Delta_2}{\partial Z} - \bar{w}_{14}\frac{\partial^2 q_0}{\partial Z^2}, \qquad (6.38)$$

where the bar notation indicates the mean part of the relevant expression. Integration of this equation shows that, for impermeability at both boundaries, we must satisfy the linear partial differential equation

$$\frac{\partial^2 q_0}{\partial X^2} I_1 + \frac{\partial^2 q_0}{\partial Z^2} I_2 = \frac{\partial \Delta_1}{\partial X}\frac{\partial \kappa_1}{\partial \alpha_0} I_1 + \frac{\partial \Delta_2}{\partial Z}\frac{\partial \kappa_2}{\partial \beta_0} I_2, \qquad (6.39)$$

where

$$I_1 = \int_{y=-1}^{+1} \bar{u}_{13}\,\mathrm{d}y \quad \text{and} \quad I_2 = \int_{y=-1}^{+1} \bar{w}_{14}\,\mathrm{d}y. \tag{6.40}$$

Now returning to the solvability condition determining Ω_2, we can see that further simplifications are possible by introducing some particular solutions; for example,

$$\bar{L}_u\left\{\frac{\partial u_{11}}{\partial \kappa_1}, \frac{\partial v_{11}}{\partial \kappa_1}, \frac{\partial p_{11}}{\partial \kappa_1}\right\} = S_1, \tag{6.41}$$

$$\bar{L}_u\left\{\frac{\partial u_{13}}{\partial \kappa_1}, \frac{\partial v_{13}}{\partial \kappa_1}, \frac{\partial p_{13}}{\partial \kappa_1}\right\} = S_2, \tag{6.42}$$

$$\bar{L}_u\left\{\frac{\partial u_{12}}{\partial \kappa_2}, \frac{\partial v_{12}}{\partial \kappa_2}, \frac{\partial p_{12}}{\partial \kappa_2}\right\} = S_3, \tag{6.43}$$

where S_1, S_2, S_3 are the $[\cdots]$ coefficients of the $\Delta_1 q_{0X}$, $(q_{0X})^2$ and $\Delta_2 q_{0Z}$ terms, respectively, in (6.36). These particular solutions can be applied to the system as a whole, reducing the form of the frequency correction term to

$$\Omega_2 = \Omega_{21}\Delta_1^2 + \Omega_{22}\Delta_2^2 + \Phi_1\frac{\partial \Delta_1}{\partial X} + \Phi_2\frac{\partial \Delta_2}{\partial Z} + \Phi_3\frac{\partial^2 q_0}{\partial X^2}, \tag{6.44}$$

where $\{\Omega_{2i}, \Phi_i\}$ are constants determined from the appropriate integrals contained in the solvability condition. Furthermore, a perturbation of $\{\alpha_0, \beta_0\}$ in the leading-order problem shows that

$$\Omega_{21} = \tfrac{1}{2}\frac{\partial^2 \Omega_0}{\partial \alpha_0^2}, \quad \Omega_{22} = \tfrac{1}{2}\frac{\partial^2 \Omega_0}{\partial \beta_0^2}. \tag{6.45}$$

The consistency conditions now determine the evolution of the wavenumber perturbations, and applying the above results we obtain

$$\delta\frac{\partial \Delta_1}{\partial T} + \delta\frac{\partial \Omega_0}{\partial \alpha_0}\frac{\partial \Delta_1}{\partial X} + \delta^2\left\{\frac{\partial^2 \Omega_0}{\partial \alpha_0^2}\Delta_1\frac{\partial \Delta_1}{\partial X} + \frac{\partial^2 \Omega_0}{\partial \beta_0^2}\Delta_2\frac{\partial \Delta_2}{\partial X} + \Phi_1\frac{\partial^2 \Delta_1}{\partial X^2} + \Phi_2\frac{\partial^2 \Delta_2}{\partial X \partial Z} + \Phi_3\frac{\partial^3 q_0}{\partial X^3}\right\} + O(\delta^3) = 0, \tag{6.46}$$

$$\frac{\partial \Delta_1}{\partial Z} = \frac{\partial \Delta_2}{\partial X}. \tag{6.47}$$

Now for a leading-order balance we again introduce the slower time-scale τ (defined by (2.43)) and make the Galilean transformation (2.44). Now by defining $\tilde{\Theta}$ as

$$\frac{\partial \tilde{\Theta}}{\partial \xi} = \Delta_1, \quad \frac{\partial \tilde{\Theta}}{\partial Z} = \Delta_2, \tag{6.48}$$

we obtain

$$\tilde{\Theta}_{\xi\tau} + \frac{\partial^2 \Omega_0}{\partial \alpha_0^2}\tilde{\Theta}_\xi\tilde{\Theta}_{\xi\xi} + \frac{\partial^2 \Omega_0}{\partial \beta_0^2}\tilde{\Theta}_Z\tilde{\Theta}_{\xi Z} = -\Phi_1\tilde{\Theta}_{\xi\xi\xi} - \Phi_2\tilde{\Theta}_{\xi ZZ} - \Phi_3 q_{0\xi\xi\xi} \tag{6.49}$$

and

$$q_{0\xi\xi} + \lambda_1 q_{0ZZ} = -\lambda_2\tilde{\Theta}_{\xi\xi} - \lambda_3\tilde{\Theta}_{ZZ}. \tag{6.50}$$

As mentioned above, the terms $\{\Phi_1, \Phi_2, \Phi_3\}$ can be calculated numerically (although it is not a trivial computation) from the integrals derived in the solvability condition, and $\{\lambda_1, \lambda_2, \lambda_3\}$ denote the coefficients

$$\left\{\frac{I_2}{I_1}, \frac{\partial \kappa_1}{\partial \alpha_0}, \frac{I_2}{I_1}\frac{\partial \kappa_2}{\partial \beta_0}\right\}$$

in (6.39). We observe that if $\partial_Z \equiv 0$, then $q_{0\xi\xi\xi}$ reduces to a constant multiple of $\Delta_{1\xi\xi}$, the $O(\delta)$ problem is simplified, and we return to the Burgers equation once more.

There is little we can say about the solution of this equation without a detailed numerical investigation and computation of the coefficients from their integral representations. The form of the system (6.49)–(6.50) can be simplified by rescaling appropriately but only to the form

$$A_{\chi\tau} + A_\chi A_{\chi\chi} \pm A_\zeta A_{\zeta\chi} = \pm A_{\chi\chi\chi} + c_1 A_{\chi\zeta\zeta} + B_{\chi\chi\chi}, \tag{6.51}$$

$$B_{\chi\chi} + c_2 B_{\zeta\zeta} = c_2 A_{\chi\chi} + c_3 A_{\zeta\zeta}, \tag{6.52}$$

where c_j are constants, which can be written in terms of those appearing previously. We note that the first equation, (6.51), can be integrated with respect to χ if we redefine the dependent variable to include an arbitrary function of τ. We also note (as in Davey *et al*. 1974) that a return to the Burgers equation (after a further rescaling) is achieved for skewed two-dimensional perturbations; that is, for wavenumber perturbations which may be written solely in terms of a skewed variable,

$$\Xi = l\chi + m\zeta, \tag{6.53}$$

for some real values l and m. Obviously, the sign of the diffusive term will determine the type of solution obtained, but this must again be calculated from the coefficients introduced above and any chosen values of $\{l, m\}$.

7. Conclusion

We have applied a phase equation technique to develop a perturbation theory for slowly varying finite-amplitude wavetrains, at finite Reynolds numbers, in plane Poiseuille flow. When the method is applied to uniform wavetrains we have shown that a small wavenumber perturbation evolves according to the Burgers equation. We can obtain an exact solution to the Burgers equation via the Cole–Hopf transformation and the solution is characterized by the sign of the diffusive coefficient. For a typical uniform wavetrain with $O(1)$ disturbance energy, we have discussed how to compute the coefficients appearing in the evolution equation.

A number of numerical results, for solutions corresponding to cross-sections of the neutral surface, have been presented showing that both diffusively stable and unstable cases are possible in general. In the unstable case (Φ positive), we know that the wavenumber perturbation develops a singularity at finite time in the slow scale. For the diffusively stable case (Φ negative), it is possible for weak shock structures to appear for non-localized or non-periodic initial conditions.

Interpreting these results physically, we see that finite-amplitude uniform wavetrain solutions should not be observed in plane Poiseuille flow. Obviously, for this flow configuration, we must also note that this is only one of a number of active instability mechanisms. We make no claims that the modulational instability of the wavetrain is in any way the 'most unstable', but since an initial disturbance can always be found that does not decay (for two-dimensional waves), it suggests that the use of uniform wavetrains in theory and computation needs some justification for large-scale problems. We can only say at this stage that the uniform wavetrains are unstable through this mechanism (on the previously defined slow spatial and temporal scales), and not what the large-time form of a wavenumber perturbation would be. As the wavenumber perturbation either develops shock discontinuities or a finite-time singularity we must at some stage reintroduce previously neglected terms into the evolution equation, (2.47), to give a new form valid near the shock or at the time of breakdown as appropriate.

As the shock structures in the wavenumber perturbation are approached, we expect higher-order spatial derivatives to become of equal importance in the evolution equation. We see (as noted by Bernoff 1988), by an induction argument, that higher-order frequency corrections are of the form

$$\Omega_n = P_n\left(\frac{\partial \alpha}{\partial X}, \frac{\partial^2 \alpha}{\partial X^2}, \dots, \frac{\partial^n \alpha}{\partial X^n}\right), \tag{7.1}$$

a polynomial in the slow-scale derivatives of the wavenumber. Thus as the length-scales shorten, all the higher-order terms will become of comparable magnitude simultaneously. When all the previously neglected terms return to the evolution equation, we must return to the full unsteady two-dimensional Navier–Stokes equations to determine the development.

We have shown in § 3 that the phase equation method is distinct from a multi-scale amplitude perturbation approach. A perturbation of the phase in the form,

$$\theta = \theta_0(x,t) + (\epsilon/\delta)\Theta_1(\delta x, \delta t) + \cdots, \tag{7.2}$$

may be expected to be equivalent to an amplitude perturbation for ϵ sufficiently small. For uniform wavetrains, however, we showed, in § 2, that for a leading-order balance we required that the wavenumber perturbation be of comparable magnitude to the slow-scale parameter, thus $\epsilon \sim \delta$ and the multi-scale approach must be distinct. The evolution equation obtained from the amplitude perturbation method is simply the heat transfer equation and no nonlinearity can be introduced. For the diffusively unstable case, we must rely on linear higher-order derivatives to become of comparable magnitude and alter the form of the evolution equation.

We have also shown that the evolution equation must break down to a different form as the leading-order problem approaches the linear limit or the region where the upper and lower branches join at finite amplitude. This singularity is due to the behaviour of the neutral surface in these areas, and in the weakly nonlinear case we observed that a regularization of the evolution equation is non-trivial. Indeed this has to be the case if we are to be able to reconcile the phase equation approach with the well-known results of weakly nonlinear theory, namely the Stuart–Landau equation.

To investigate the effects of three dimensionality we also developed a form for the evolution equation by extending the definitions of the phase method applied in previous sections. To simplify the basic analysis a *weak* spanwise dependence is allowed for in the uniform wavetrain problem; the analogous results for a fully three-dimensional situation are stated in the appendix. In this case we show that the evolution equation is more complex, and is coupled with a further linear partial differential equation. This coupled equation determines an additional pressure function (required in order to satisfy an impermeability condition at the parallel boundaries) that in the two-dimensional formulation is formally eliminated by using a stream function approach. The comments made above concerning the breakdown of the phase equation method, at the $E = 0$ plane of the neutral surface, will similarly apply here in the three-dimensional approach. We should expect to be able to, likewise, match the phase method, in the appropriate areas of the neutral surface, to the results of Davey *et al*. (1974) concerning the weakly nonlinear development of three-dimensional disturbances. The multi-scale approach, of § 3, can be similarly extended to three dimensions, giving the heat equation with diffusion on both slow scales although this is again linearly coupled with the same partial differential equation for the pressure term.

Thus, generally, the development of, and role played by, these uniform wavetrains in PPF is complex to determine. We have shown susceptibility to weak shocks and finite-time singularities for both upper and lower branch solutions, together with a complex regularization problem in the $E \to 0$ limit. As well as these slow-scale effects, we also have the superharmonic instability results of Pugh & Saffman (1988), which occur on $O(1)$ scales, the apparent slow decay of three-dimensional perturbations to the fully developed flow as discussed by Orszag & Patera (1983) and the bypass mechanisms of Gustavsson *et al.*, to mention just some of the instability mechanisms present. Obviously, it is difficult to say (in any given problem) how such mechanisms will interact but to assume that any one will dominate over others needs to be carefully justified.

There are a number of questions that remain unanswered in this discussion, i.e. matching the phase equation theory to a relevant approach in the weakly nonlinear case, and the effect of the weak three dimensionality through computation of the evolution equations (6.49)–(6.50). If in future we wish to produce a full numerical procedure, to give refined values for the neutral surface, or for computation of the evolution equation with weak spanwise dependence, we should perhaps consider replacing the numerical method described previously with a similar collocation method. Such techniques were used successfully at a later stage by Herbert (1976) and have the advantage of reducing computation time by allowing for a simpler evaluation of the nonlinear terms in Newton's method.

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Appendix A. The fully three-dimensional problem

In this appendix we present some of the more lengthy details involved with the application of phase equation methods to a fully three-dimensional problem; by which we mean (in this application) the modulation over long spanwise and streamwise scales of a fully oblique uniform wavetrain in PPF. We prefer to present these details as an appendix in order to keep the basic method (as presented in § 6 for the case of a weak spanwise dependence) as simple as possible.

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Substituting the form of solution for the $O(\delta)$ system, defined by (6.25)–(6.28), into the inhomogeneous terms at $O(\delta^2)$ yields

$$\underline{\boldsymbol{L}}\{u_2, v_2, w_2, p_2\} = \underline{\boldsymbol{R}}_1 \Delta_1^2 + \underline{\boldsymbol{R}}_2 \Delta_2^2 + \underline{\boldsymbol{R}}_3 \Delta_1 \Delta_2 + \underline{\boldsymbol{R}}_4 \frac{\partial \Delta_1}{\partial X} + \underline{\boldsymbol{R}}_5 \frac{\partial \Delta_2}{\partial X} + \underline{\boldsymbol{R}}_6 \frac{\partial \Delta_1}{\partial Z} + \underline{\boldsymbol{R}}_7 \frac{\partial \Delta_2}{\partial Z} + \underline{\boldsymbol{R}}_8 \frac{\partial^2 q_0}{\partial X^2} + \underline{\boldsymbol{R}}_9 \frac{\partial^2 q_0}{\partial X \partial Z} + \underline{\boldsymbol{R}}_{10} \frac{\partial^2 q_0}{\partial Z^2} + \underline{\boldsymbol{R}}_{11} \Delta_1 \frac{\partial q_0}{\partial X} + \underline{\boldsymbol{R}}_{12} \Delta_2 \frac{\partial q_0}{\partial X} + \underline{\boldsymbol{R}}_{13} \Delta_1 \frac{\partial q_0}{\partial Z} + \underline{\boldsymbol{R}}_{14} \Delta_2 \frac{\partial q_0}{\partial Z} + \underline{\boldsymbol{R}}_{15} \left(\frac{\partial q_0}{\partial X}\right)^2 + \underline{\boldsymbol{R}}_{16} \frac{\partial q_0}{\partial X}\frac{\partial q_0}{\partial Z} + \underline{\boldsymbol{R}}_{17} \left(\frac{\partial q_0}{\partial Z}\right)^2 + \Omega_2 \frac{\partial \hat{\underline{\boldsymbol{u}}}_0}{\partial \theta} - \left(\frac{\partial q_1}{\partial X}, 0, \frac{\partial q_1}{\partial Z}\right)^{\mathrm{T}}, \tag{A 1}$$

as the vector form of the momentum equations, where $\underline{\boldsymbol{L}} \equiv (L_u, L_v, L_w)^{\mathrm{T}}$, together with a continuity of mass condition,

$$\alpha_0 \frac{\partial u_2}{\partial \theta} + \frac{\partial v_2}{\partial y} + \beta_0 \frac{\partial w_2}{\partial \theta} = -\left\{ \frac{\partial u_{11}}{\partial \theta} \Delta_1^2 + \frac{\partial w_{12}}{\partial \theta} \Delta_2^2 + \frac{\partial}{\partial \theta}(u_{12} + w_{11}) \Delta_1 \Delta_2 + u_{11} \frac{\partial \Delta_1}{\partial X} + u_{12} \frac{\partial \Delta_2}{\partial X} + w_{11} \frac{\partial \Delta_1}{\partial Z} + w_{12} \frac{\partial \Delta_2}{\partial Z} + u_{13} \frac{\partial^2 q_0}{\partial X^2} + (u_{14} + w_{13}) \frac{\partial^2 q_0}{\partial X \partial Z} + w_{14} \frac{\partial^2 q_0}{\partial Z^2} + \frac{\partial u_{13}}{\partial \theta} \Delta_1 \frac{\partial q_0}{\partial X} + \frac{\partial w_{13}}{\partial \theta} \Delta_2 \frac{\partial q_0}{\partial X} + \frac{\partial u_{14}}{\partial \theta} \Delta_1 \frac{\partial q_0}{\partial Z} + \frac{\partial w_{14}}{\partial \theta} \Delta_2 \frac{\partial q_0}{\partial Z} \right\}. \tag{A 2}$$

The coefficients, denoted by $\underline{\boldsymbol{R}}_i$ above, are straightforward to determine; for example,

$$\underline{\boldsymbol{R}}_1 = \frac{\partial \Omega_0}{\partial \alpha_0} \frac{\partial \underline{\boldsymbol{u}}_{11}}{\partial \theta} - \hat{u}_0 \frac{\partial \underline{\boldsymbol{u}}_{11}}{\partial \theta} - \alpha_0 u_{11} \frac{\partial \underline{\boldsymbol{u}}_{11}}{\partial \theta} - u_{11} \frac{\partial \hat{\underline{\boldsymbol{u}}}_0}{\partial \theta} - v_{11} \frac{\partial \underline{\boldsymbol{u}}_{11}}{\partial y} - \beta_0 w_{11} \frac{\partial \underline{\boldsymbol{u}}_{11}}{\partial \theta} - \left(\frac{\partial p_{11}}{\partial \theta}, 0, 0\right)^{\mathrm{T}} + \frac{1}{Re}\left\{2\alpha_0 \frac{\partial^2 \underline{\boldsymbol{u}}_{11}}{\partial \theta^2} + \frac{\partial^2 \hat{\underline{\boldsymbol{u}}}_0}{\partial \theta^2}\right\}. \tag{A 3}$$

This system requires a solvability condition to be satisfied and therefore determines Ω_2 for a given leading-order solution. Before computing the integrals needed for the evaluation of Ω_2 we note that a number of the inhomogeneous terms given above can be removed from the solvability condition by introducing some particular solutions,

$$\underline{\boldsymbol{L}}\left\{\frac{\partial u_{11}}{\partial \kappa_1}, \frac{\partial v_{11}}{\partial \kappa_1}, \frac{\partial w_{11}}{\partial \kappa_1}, \frac{\partial p_{11}}{\partial \kappa_1}\right\} = \underline{\boldsymbol{R}}_{11}, \quad \underline{\boldsymbol{L}}\left\{\frac{\partial u_{12}}{\partial \kappa_1}, \frac{\partial v_{12}}{\partial \kappa_1}, \frac{\partial w_{12}}{\partial \kappa_1}, \frac{\partial p_{12}}{\partial \kappa_1}\right\} = \underline{\boldsymbol{R}}_{12}, \tag{A 4}$$

$$\underline{\boldsymbol{L}}\left\{\frac{\partial u_{11}}{\partial \kappa_2}, \frac{\partial v_{11}}{\partial \kappa_2}, \frac{\partial w_{11}}{\partial \kappa_2}, \frac{\partial p_{11}}{\partial \kappa_2}\right\} = \underline{\boldsymbol{R}}_{13}, \quad \underline{\boldsymbol{L}}\left\{\frac{\partial u_{12}}{\partial \kappa_2}, \frac{\partial v_{12}}{\partial \kappa_2}, \frac{\partial w_{12}}{\partial \kappa_2}, \frac{\partial p_{12}}{\partial \kappa_2}\right\} = \underline{\boldsymbol{R}}_{14}, \tag{A 5}$$

$$\underline{\boldsymbol{L}}\left\{\frac{\partial u_{13}}{\partial \kappa_1}, \frac{\partial v_{13}}{\partial \kappa_1}, \frac{\partial w_{13}}{\partial \kappa_1}, \frac{\partial p_{13}}{\partial \kappa_1}\right\} = \underline{\boldsymbol{R}}_{15}, \quad \underline{\boldsymbol{L}}\left\{\frac{\partial u_{14}}{\partial \kappa_1}, \frac{\partial v_{14}}{\partial \kappa_1}, \frac{\partial w_{14}}{\partial \kappa_1}, \frac{\partial p_{14}}{\partial \kappa_1}\right\} = \underline{\boldsymbol{R}}_{16}, \tag{A 6}$$

$$\underline{\boldsymbol{L}}\left\{\frac{\partial u_{14}}{\partial \kappa_2}, \frac{\partial v_{14}}{\partial \kappa_2}, \frac{\partial w_{14}}{\partial \kappa_2}, \frac{\partial p_{14}}{\partial \kappa_2}\right\} = \underline{\boldsymbol{R}}_{17}, \tag{A 7}$$

together with the previous solutions (6.25)–(6.28); obviously these solutions likewise apply to the appropriate inhomogeneous terms of the continuity equation.

The explicit content of the solvability condition can be determined by computing the adjoint and relevant integrals; although we do not give such details here it is noted that the final form of the frequency correction must be

$$\Omega_2 = \hat{\Omega}_{21}\Delta_1^2 + \hat{\Omega}_{22}\Delta_1\Delta_2 + \hat{\Omega}_{23}\Delta_2^2 + \hat{\Phi}_1\frac{\partial \Delta_1}{\partial X} + \hat{\Phi}_2\frac{\partial \Delta_2}{\partial X} + \hat{\Phi}_3\frac{\partial \Delta_1}{\partial Z} + \hat{\Phi}_4\frac{\partial \Delta_2}{\partial Z} + \hat{\Phi}_5\frac{\partial^2 q_0}{\partial X^2} + \hat{\Phi}_6\frac{\partial^2 q_0}{\partial X \partial Z} + \hat{\Phi}_7\frac{\partial^2 q_0}{\partial Z^2}, \quad \text{(A 8)}$$

for coefficients Ω_{2i} and $\hat{\Phi}_i$, which, in general, must be determined numerically. Although we can not remove the terms Δ_1^2, $\Delta_1\Delta_2$, Δ_2^2 from the solvability condition, we can effectively remove them from the computation by noting that a further wavenumber perturbation of the systems satisfied by $\{u_{11}, v_{11}, w_{11}, p_{11}\}$ and $\{u_{12}, v_{12}, w_{12}, p_{12}\}$ yields

$$\hat{\Omega}_{21} = \frac{1}{2}\frac{\partial^2 \Omega_0}{\partial \alpha_0^2}, \quad \hat{\Omega}_{22} = \frac{\partial^2 \Omega_0}{\partial \alpha_0 \partial \beta_0}, \quad \hat{\Omega}_{23} = \frac{1}{2}\frac{\partial^2 \Omega_0}{\partial \beta_0^2}. \quad \text{(A 9)}$$

The evolution of the wavenumber perturbations is then governed by the following system:

$$\frac{\partial \Delta_1}{\partial \tau} + \frac{\partial^2 \Omega_0}{\partial \alpha_0^2}\Delta_1\frac{\partial \Delta_1}{\partial \xi} + \frac{\partial^2 \Omega_0}{\partial \beta_0^2}\Delta_2\frac{\partial \Delta_2}{\partial \xi} + \frac{\partial^2 \Omega_0}{\partial \alpha_0 \partial \beta_0}\frac{\partial}{\partial \xi}(\Delta_1\Delta_2) = -\hat{\Phi}_1\frac{\partial^2 \Delta_1}{\partial \xi^2} - \hat{\Phi}_2\frac{\partial^2 \Delta_2}{\partial \xi^2} - \hat{\Phi}_3\frac{\partial^2 \Delta_1}{\partial \zeta \partial \xi} - \hat{\Phi}_4\frac{\partial^2 \Delta_2}{\partial \zeta \partial \xi} - \hat{\Phi}_5\frac{\partial^3 q_0}{\partial \xi^3} - \hat{\Phi}_6\frac{\partial^3 q_0}{\partial \zeta \partial \xi^2} - \hat{\Phi}_7\frac{\partial^3 q_0}{\partial \xi \partial \zeta^2}, \quad \text{(A 10)}$$

$$\frac{\partial \Delta_1}{\partial \zeta} = \frac{\partial \Delta_2}{\partial \xi}, \quad \text{(A 11)}$$

$$\frac{\partial^2 q_0}{\partial \xi^2} + \hat{\lambda}_1\frac{\partial^2 q_0}{\partial \xi \partial \zeta} + \hat{\lambda}_2\frac{\partial^2 q_0}{\partial \zeta^2} = -\hat{\lambda}_3\frac{\partial \Delta_1}{\partial \xi} - \hat{\lambda}_4\frac{\partial \Delta_2}{\partial \xi} - \hat{\lambda}_5\frac{\partial \Delta_1}{\partial \zeta} - \hat{\lambda}_6\frac{\partial \Delta_2}{\partial \zeta}, \quad \text{(A 12)}$$

where we have introduced the constants $\hat{\lambda}_i$, the slower time-scale τ and the new coordinate ζ defined by

$$\zeta = Z - \frac{\partial \Omega_0}{\partial \beta_0}T. \quad \text{(A 13)}$$

The equation (A 12) and constants $\hat{\lambda}_i$ are obtained from solving the continuity equation (A 2) for the mean-flow term, $\bar{v}_2$, and applying the impermeability condition at the boundaries $y = \pm 1$.

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